

A Note on the Convergence of Godunov Type Methods for Shock Reflection Problems

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Abstract

This paper identifies a new pathology that can be found for numerical simulations of nonlinear conservation law systems. Many of the difficulties already identified in the literature (rarefaction shocks, carbuncle phenomena, slowly moving shocks, wall heating, etc) can be traced to insufficient numerical dissipation, and the current case is no different. However, the details of the case we study here are somewhat unique in that the solution which is found by the numerics can fail to have a derivative anywhere in the post-shock region.

1 Introduction

Theoretical convergence characteristics of numerical methods for systems of nonlinear PDEs have been difficult to ascertain even in 1D. With the notable exception of the random choice method of Glimm [1] and its extensions, rigorous error bounds have remained largely elusive. This is not a situation where there is simply a hole in the theory but convergence troubles are never found in practice. There are in fact a number of well-known examples where numerical methods are known to behave poorly. Examples in a single space dimension include rarefaction shocks at sonic points [2], the so-called wall heating phenomenon [3, 4], and sub-linear convergence for linear waves [5, 6]. Often these difficulties are associated with discontinuous solutions or a lack of sufficient dissipation in the method because the nonlinear artificial dissipation inherent to the schemes vanishes at certain points in the flow. We investigate here a new pathology where the dissipation is insufficient over a large portion of the domain. The result is convergence to a weak solution which is nowhere differentiable. The exact nature of this weak solution is seen to depend heavily on the choice of time step. Admittedly this poor behavior could be eliminated with a simple linear artificial viscosity, but the intent here is simply to indicate the kind of behavior that can be found.

2 Governing equations and model problem

Consider the one-dimensional Euler equations with ideal equation of state

$$\frac{\partial}{\partial t} \mathbf{u} + \frac{\partial}{\partial x} \mathbf{f}(\mathbf{u}) = 0, \quad (1)$$

where $\mathbf{u} = [\rho, \rho u, \rho E]^T$ and $\mathbf{f}(\mathbf{u}) = [\rho u, \rho u^2 + p, u(\rho E + p)]^T$. Here ρ is the density, u the velocity, E the total energy per unit mass, and p the pressure. The total energy for the fluid is given by $E = e + \frac{1}{2}u^2$ where the equation of state is given by $e = \frac{p}{\rho(\gamma-1)}$ with γ the ratio of specific heats.

We investigate a shock reflection problem with $\gamma = 1.4$ on the domain $x \in [-.5, .5]$. The initial conditions in primitive variables are

$$[\rho, u, p] = \begin{cases} \left[1.0, 2.0, \frac{1}{\gamma} \right] & \text{for } x < 0 \\ \left[1.0, -2.0, \frac{1}{\gamma} \right] & \text{for } x \geq 0. \end{cases}$$

Inflow conditions are applied at domain boundaries, and we integrate to time $t_f = 0.5$. The resulting solution consists of two approximately Mach 2.76205 shocks moving outward from the origin. The quantitative nature

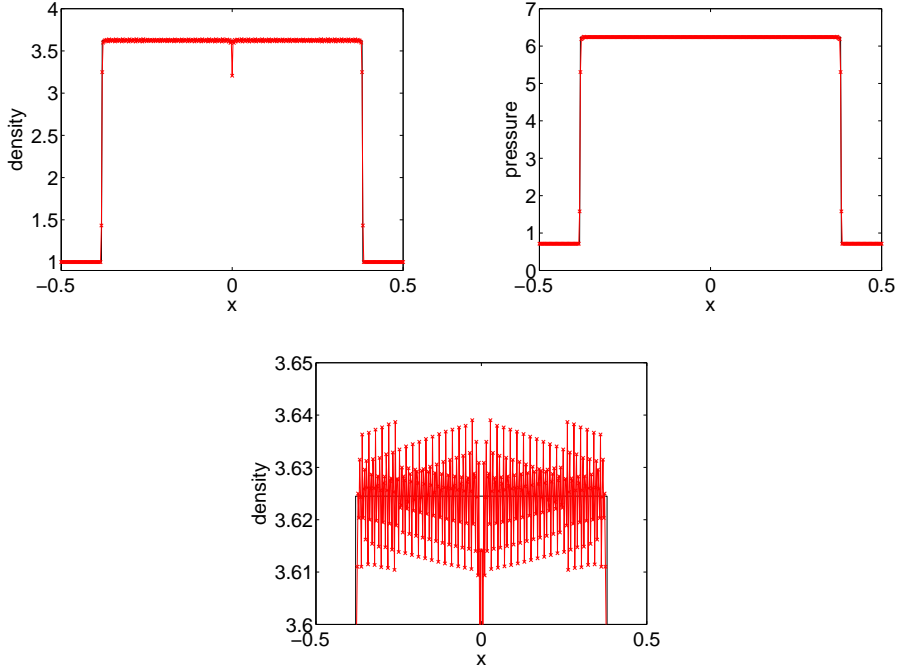


Figure 1: Density (top left), pressure (top right), and a zoom of the density (bottom). In all plots the black line is the entropy satisfying solution and the red 'x' marks and corresponding line are a first-order Godunov type approximation.

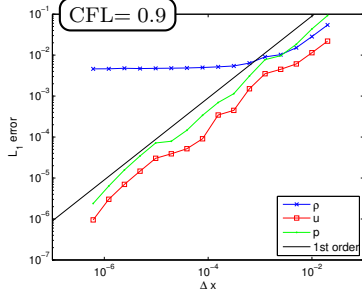
of the phenomenon studied here generally scales with the Mach number. However, the qualitative conclusions remain unchanged as Mach number is varied and therefore in the interest of brevity we present simulation results for only one case.

3 Numerical results

We begin by approximating the solution using a first-order Godunov type method [7] with Roe's approximate Riemann solver [8, 9]. Note that the results do not change in any significant way if one instead uses an exact Riemann solver. Discretization is performed on the computational mesh $x_i = -0.5 + (i-1)\Delta x$ for $i = 1 \dots m$ and $\Delta x = 1/(m-1)$. Initial conditions are applied with exact states to the left and right of the origin. For the cases considered here, $x_i \neq 0$ for any i and the initial condition is applied as an exact conservative average of the left and right conservative states.

Approximate solutions to the shock reflection problem are shown in Figure 1. Shown are the density, pressure, and a restricted view of the density for a CFL number of 0.9 and $m = 401$. The eventual limiting behavior can already be seen in the zoom of the density where the solution oscillates around the entropy satisfying solution. Because the artificial viscosity in Godunov type methods is dependent on the velocity and because the exact solution has no post-shock velocity for this problem, the magnitude of the oscillations does not decrease if the post-shock velocity converges to zero fast enough in some sense. This appears to be the case and the frequency of oscillation increases in an unbounded manner as the grid resolution increases. Such an approximation will be correct in some average sense, but will not converge in an L_2 or even L_1 sense. The solution to which the numerical approximation is converging appears to be a measure valued solution, whose value at a point can be drawn from a statistical distribution. Intuitively one can think that as $\Delta x \rightarrow 0$, the approximate solution in the post-shock region is converging to a solution that lives in a band containing the exact solution. The size of that band is dependent on the details of the discretization, and most prominently the time step. Furthermore the width can be zero for certain circumstances, as shown below.

We perform a convergence study using the discrete L_1 norm to judge convergence. The results are shown in the table of Figure 2. We can see that the density does not converge below approximately 5×10^{-3} . To



m	$e_\rho(m)$	κ	$e_u(m)$	κ	$e_p(m)$	κ
51	5.444e-02	—	2.199e-02	—	9.267e-02	—
101	2.838e-02	0.94	1.146e-02	0.94	4.357e-02	1.09
201	1.517e-02	0.90	6.098e-03	0.91	1.841e-02	1.24
401	1.028e-02	0.56	4.502e-03	0.44	9.670e-03	0.93
801	9.064e-03	0.18	3.516e-03	0.36	7.760e-03	0.32
1601	6.375e-03	0.51	1.509e-03	1.22	3.084e-03	1.33
3201	5.438e-03	0.23	4.491e-04	1.75	1.146e-03	1.43
6401	5.185e-03	0.07	3.460e-04	0.38	6.985e-04	0.71
12801	4.983e-03	0.06	9.070e-05	1.93	3.398e-04	1.04
25601	4.864e-03	0.03	5.198e-05	0.80	1.468e-04	1.21
51201	4.804e-03	0.02	3.907e-05	0.41	7.907e-05	0.89
102401	4.764e-03	0.01	3.029e-05	0.37	7.161e-05	0.14
204801	4.707e-03	0.02	1.473e-05	1.04	3.452e-05	1.05
409601	4.778e-03	-0.02	6.984e-06	1.17	1.575e-05	1.13
819201	4.644e-03	0.04	3.031e-06	1.20	6.367e-06	1.31
1638401	4.628e-03	0.005	9.497e-07	1.67	2.387e-06	1.42

Figure 2: L_1 errors and convergence rates for density, velocity, and pressure for CFL=0.9. A reference line indicating first-order convergence is also plotted.

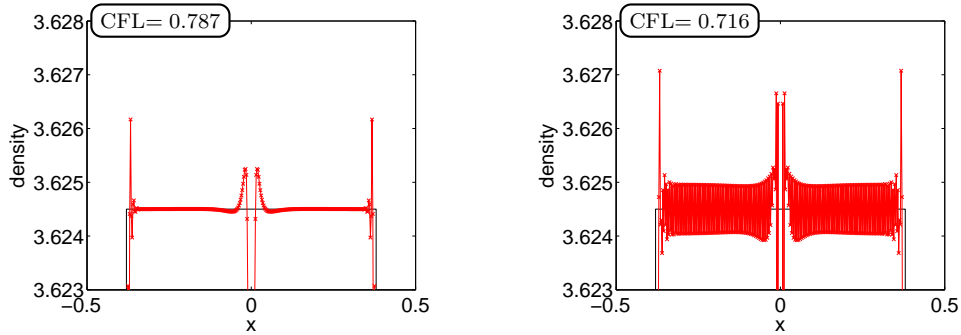
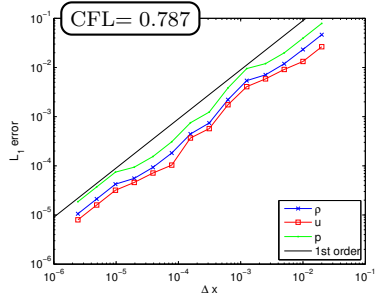


Figure 3: Zoom of the density where the shock travels through one computational cell in exactly 5 time steps (left) and exactly 5.5 steps (right).

understand the details, consider a single cell as it transitions from before to after the shock. Because the method is conservative and seems to be converging to some weak solution, the approximate shock location should be correct in the limit. However, each point travels through the shock in a slightly different manner which leads to the oscillations in density. To see this more clearly we can modify the time step such that the shock travels through a computational cell in an integral number of time steps. It is determined that the shock speed is $\approx .76205$. For the results in Figure 1 (with CFL= 0.9 and $m = 401$), the time step is found to be $\Delta t \approx 7.496 \times 10^{-4}$. As a result, the shock travels through each cell in approximately 4.3765 time steps. For the case of $m = 401$, a time step of $\Delta t \approx 6.5613 \times 10^{-4}$ corresponding to CFL ≈ 0.787 , will have the shock traveling through each computational cell in exactly 5 time steps. Figure 3 demonstrates that this choice does indeed remove the post-shock density oscillations. Also in this figure we show the results for $\Delta t = 5.965 \times 10^{-4}$ corresponding to CFL ≈ 0.716 where the shock travels through one cell in exactly 5.5 time steps. Here we see that every other cell travels through the shock in the same way with the result that the density oscillates cell to cell between two values. The table in Figure 4 shows the results of a grid convergence study with CFL ≈ 0.787 where the shock travels through each cell in exactly 5 time steps. Here the density is seen to converge well even to the finest resolution. The overall conclusions remain unchanged and convergent approximations are found for other CFL numbers that correspond to the shock traversing a cell in an integral numbers of time steps. In particular we have tested 4 through 10 and the arbitrary value 37 (below 4 the CFL restriction is violated). This is a purely discrete phenomenon where convergence occurs for CFL numbers that result in a shock that traverses a computational cell in an integral number of time steps. Other CFL numbers will in general produce oscillatory results and nonconvergence where the

L_1 norm of the density will converge only to a level that depends on the CFL number and other simulation parameters.



m	$e_\rho(m)$	κ	$e_u(m)$	κ	$e_p(m)$	κ
51	4.646e-02	—	2.649e-02	—	7.896e-02	—
101	2.325e-02	1.00	1.327e-02	1.00	3.945e-02	1.00
201	1.191e-02	0.97	9.133e-03	0.54	1.984e-02	0.99
401	7.097e-03	0.75	5.873e-03	0.64	1.201e-02	0.72
801	5.412e-03	0.39	4.078e-03	0.53	9.524e-03	0.33
1601	2.217e-03	1.29	1.746e-03	1.22	3.839e-03	1.31
3201	7.444e-04	1.57	5.706e-04	1.61	1.240e-03	1.63
6401	4.436e-04	0.75	3.671e-04	0.64	7.506e-04	0.72
12801	1.818e-04	1.29	1.036e-04	1.82	3.081e-04	1.28
25601	9.305e-05	0.97	7.133e-05	0.54	1.549e-04	0.99
51201	5.545e-05	0.75	4.589e-05	0.64	9.383e-05	0.72
102401	4.228e-05	0.29	3.186e-05	0.53	7.441e-05	0.33
204801	2.114e-05	1.00	1.593e-05	1.00	3.720e-05	1.00
409601	1.057e-05	1.00	7.965e-06	1.00	1.860e-05	1.00

Figure 4: L_1 errors and convergence rates for density, velocity, and pressure for time step chosen so that the shock travels through each cell in exactly 5 time steps. This corresponds to $CFL \approx 0.787$. A reference line indicating first-order convergence is also plotted.

It is important to note that the poor behavior we have been discussing is not limited to first-order discretizations. In Figure 5 we report results for convergence studies for the shock reflection problem described in Section 2 using a nominally second-order TVD discretization [10] with a MinMod limiter. Studies using $CFL = 0.9$ and $CFL \approx 0.787$ are presented¹. The exact solution of the problem under investigation consists

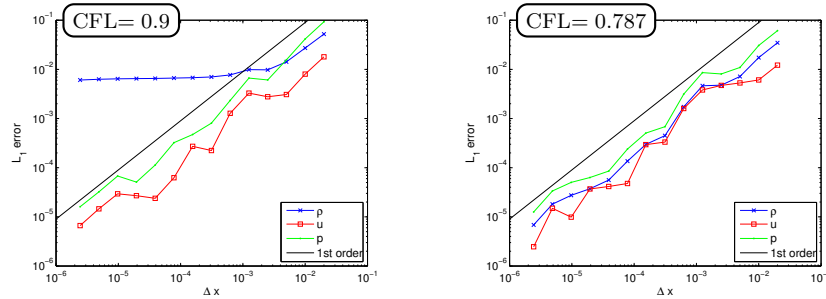


Figure 5: L_1 errors and convergence rates for density, velocity, and pressure using a TVD limited high-resolution method. Results from studies using $CFL = 0.9$ are at left and for $CFL \approx 0.787$ are at right. A reference line indicating first-order convergence is also plotted.

of only propagating shocks, and so only first-order convergence is expected. However, as was the case for the first-order methods, the TVD scheme is seen to fail to converge to the entropy satisfying solution for $CFL = 0.9$. When the time step is altered using $CFL \approx 0.787$ such that the shock traverses each cell in 5 time steps, convergence to the entropy satisfying solution is recovered.

4 Conclusions

The pathology of convergence of numerical approximations for shock reflection problems studied here is an interesting one. The specific details show some similarity to and are in some sense a mixture of the well know wall-heating phenomenon, and the phenomenon of post-shock oscillations behind slowly moving shocks. Here we showed that depending on the time step, the weak solution found by Godunov type methods can

¹Notice that the convergence character for this high-resolution scheme is somewhat less uniform than for the first-order discretizations discussed previously.

be convergent only in some average sense. The density in the post-shock region is found to depend on the number of time steps required for a shock to cross a single computational cell. By choosing the time step so that the shock crosses a cell in an integral number of time steps, pointwise convergence can be obtained everywhere except a set with zero measure. Other values of the time step result in oscillatory behavior with solutions failing to have derivatives everywhere behind the shock.

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