

1     **A STABLE ADDED-MASS PARTITIONED (AMP) ALGORITHM**  
2     **FOR ELASTIC SOLIDS AND INCOMPRESSIBLE FLOW: MODEL**  
3     **PROBLEM ANALYSIS.\***

4     D. A. SERINO<sup>†</sup>, J. W. BANKS<sup>†</sup>, W. D. HENSHAW<sup>†</sup>, AND D. W. SCHWENDEMAN<sup>†</sup>

5     **Abstract.**

6     An analysis is made of a new partitioned scheme for solving fluid-structure interaction problems  
7 involving viscous incompressible flow and compressible elastic-solids. The new scheme is stable,  
8 without sub-time-step iterations, even for light solids when added-mass and added-damping effects  
9 are large. The fluid is updated with an implicit-explicit (IMEX) fractional-step scheme whereby the  
10 velocity is advanced in one step, treating the viscous terms implicitly, and the pressure is computed  
11 in a second step. The key components of the scheme are a Robin (mixed) interface condition for  
12 the fluid pressure, and impedance based interface conditions for the velocity. While the impedance  
13 for the solid is well defined, the fluid impedance is not, and a semi-discrete local-analysis is used  
14 to inform this choice. The properties of the new scheme are analyzed and numerical results are  
15 presented to confirm the stability and accuracy of the scheme.

16     **Key words.** fluid-structure interaction, incompressible Navier-Stokes, partitioned schemes,  
17 added-mass, elastic solids

18     **AMS subject classifications.** 65M12, 74F10, 74S10, 76M20, 76D99

19     **1. Introduction.** We consider the numerical solution of fluid-structure inter-  
20 action (FSI) problems involving incompressible viscous fluids coupled to bulk elastic  
21 solids. Such problems arise in many scientific and engineering applications including  
22 flow-induced vibrations of structures (i.e., aircraft wings, undersea cables, wind tur-  
23 bines, and bridges), and blood flow in arteries and veins. FSI algorithms can either  
24 be categorized as monolithic schemes, where the numerical solutions for the fluid and  
25 solid are advanced implicitly as one large system, or partitioned schemes, where the  
26 solutions are advanced sequentially. Partitioned schemes are generally more modular  
27 and computationally efficient than monolithic schemes. However, depending on the  
28 implementation of the fluid-solid interface conditions (which need to be partitioned  
29 between the fluid and solid), partitioned schemes may suffer from instabilities, espe-  
30 cially for light solids when added-mass effects are large. Typically, partitioned schemes  
31 based on traditional Dirichlet-Neumann coupling use under-relaxed sub-time-step it-  
32 erations together with some acceleration technique to overcome instabilities (at the  
33 cost of degrading performance), see for example, [19, 21]. Reduction of sub-iterations  
34 can also be achieved by considering, for example, Robin-Neumann or Robin-Robin  
35 coupling [26, 12, 2, 22, 15, 14, 13, 16, 3, 23]. Generally the number of sub-time-step  
36 iterations increases as the solid becomes lighter, and thus it would be advantageous  
37 to have a scheme that does not require sub-iterations. The goal of our current work  
38 is to develop such a robust partitioned scheme, without the need for sub-iterations.

39     In recent work [6], we developed a new class of Added-Mass Partitioned (AMP)  
40 algorithms for FSI problems coupling incompressible flow and elastic solids that are  
41 stable without sub-iterations. The algorithms in [6] were applied to model prob-  
42 lems with infinitesimal interface displacements. The principal goals of the current

---

\*Submitted to the editors DATE.

**Funding:** This research was supported by the National Science Foundation under grants DGE-1744655, DMS-1519934, and DMS-1818926, as well as by DOE contracts from the ASCR Applied Math Program.

<sup>†</sup>Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, NY 12180, USA (serind@rpi.edu, banksj3@rpi.edu, henshw@rpi.edu, schwed@rpi.edu).

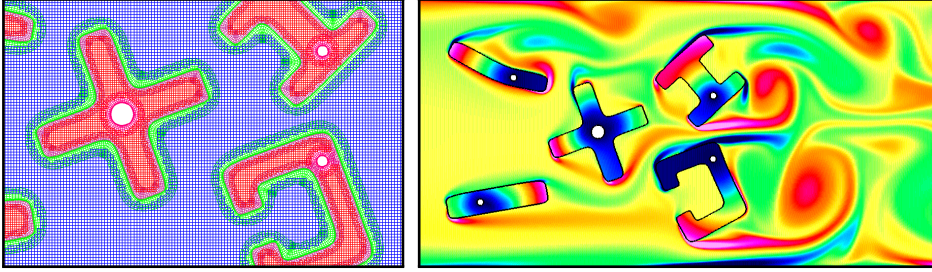


FIG. 1. *Flow past five deformable bodies in a fluid channel computed with the new AMP scheme. Left: overset grids for the fluid and solid domains. Right: contours of the vorticity in the fluid and norm of the displacement in the solids.*

43 work are to extend the original scheme to finite deformations, and to replace the ex-  
 44 plicit fluid solver with a fractional-step IMEX scheme, where the viscous terms are  
 45 treated implicitly. In the explicit scheme, the time step is chosen proportional to  
 46 the square of the grid spacing. However, the IMEX scheme has a larger stability  
 47 region so that the time step can be chosen proportional to the grid spacing. This  
 48 can be a significant advantage when the viscous terms are stiff, such as when fine  
 49 grids are used near boundaries to resolve boundary layers. The key components of  
 50 the AMP algorithm are a Robin (mixed) interface condition for the fluid pressure  
 51 and impedance-based interface conditions for the velocity. While the impedance for  
 52 the solid is well defined, the impedance for the incompressible fluid is not. A local  
 53 analysis of the semi-discrete equations for an FSI model problem is performed which  
 54 reveals that the fluid impedance has an inertial component for treating added-mass  
 55 instabilities and a viscous component for treating added-damping instabilities. The  
 56 formula for the fluid impedance is new, and its use is critical for the stability of the  
 57 AMP algorithm when the viscous CFL number becomes large and added-damping ef-  
 58 fects are significant. The properties of the new scheme are analyzed for a fundamental  
 59 FSI model problem, and numerical results are presented to confirm the stability and  
 60 accuracy of the scheme. The companion paper [24] presents the extension of the  
 61 new AMP algorithm to treat more complex configurations using overlapping grids,  
 62 such as the example shown in Figure 1. Our related work describes AMP algorithms  
 63 for other FSI problems involving incompressible viscous flow coupled to thin elastic  
 64 structures [7, 20] and rigid bodies [8, 9, 10].

65 We remark that the AMP algorithm described here was devised through numeri-  
 66 cal experimentation and the study of suitable model problems, as well as through our  
 67 experience with other FSI regimes. This strategy, based on gaining insight through  
 68 an analysis of model problems, has proved to be effective in this work and other cases,  
 69 and complements more general approaches such as those based on energy estimates.  
 70 Although the analysis presented here does not rigorously prove results for the general  
 71 FSI problem, it does provide valuable insight into the approach, as well as a justi-  
 72 fication for the success of the numerical results presented here and in [24] for more  
 73 general configurations.

74 **2. Governing equations.** We consider the coupled evolution of an incompress-  
 75 ible fluid and a linear elastic solid. The fluid occupies the domain  $\mathbf{x} \in \Omega(t)$ , where  
 76  $\mathbf{x} = (x_1, x_2, x_3)$  is a vector of physical coordinates and  $t$  is time. The velocity-pressure

77 form of the incompressible Navier-Stokes equations is given by

$$78 \quad (2.1a) \quad \rho \mathbf{v}_t + \rho(\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p = \mu \Delta \mathbf{v}, \quad \mathbf{x} \in \Omega(t),$$

$$79 \quad (2.1b) \quad \Delta p = -\rho \nabla \mathbf{v} : (\nabla \mathbf{v})^T, \quad \mathbf{x} \in \Omega(t),$$

81 where  $\mathbf{v}(\mathbf{x}, t)$  is the velocity,  $p(\mathbf{x}, t)$  is the pressure,  $\rho$  is the (constant) density, and  $\mu$   
 82 is the (constant) dynamic viscosity. The fluid stress tensor is given by  $\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\tau}$   
 83 where  $\mathbf{I}$  is the identity matrix and  $\boldsymbol{\tau}(\mathbf{x}, t) = \mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)$  is the viscous stress  
 84 tensor.

85 The equations for the solid are written in terms of the Lagrangian coordinate  
 86  $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)$  for a reference configuration  $\bar{\mathbf{x}} \in \bar{\Omega}_0$  at  $t = 0$ . (An overbar is used  
 87 here and elsewhere to denote quantities associated with the solid.) The position of the  
 88 solid in physical space is determined by the mapping  $\mathbf{x} = \bar{\mathbf{x}} + \bar{\mathbf{u}}(\bar{\mathbf{x}}, t)$ , where  $\bar{\mathbf{u}}(\bar{\mathbf{x}}, t)$  is  
 89 the displacement of the solid. The Cauchy stress tensor  $\bar{\boldsymbol{\sigma}}(\bar{\mathbf{x}}, t)$  for a linearly-elastic  
 90 solid is defined by  $\bar{\boldsymbol{\sigma}} = \bar{\lambda}(\nabla_{\bar{\mathbf{x}}} \cdot \bar{\mathbf{u}})\mathbf{I} + \bar{\mu}(\nabla_{\bar{\mathbf{x}}} \bar{\mathbf{u}} + (\nabla_{\bar{\mathbf{x}}} \bar{\mathbf{u}})^T)$ , where  $\bar{\lambda}$  and  $\bar{\mu}$  are Lamé  
 91 parameters (taken to be constants). The solid equations are considered as a first-order  
 92 system of PDEs in time and space, following [1], and are given by

$$93 \quad (2.2a) \quad \bar{\mathbf{u}}_t = \bar{\mathbf{v}}, \quad \bar{\mathbf{x}} \in \bar{\Omega}_0,$$

$$94 \quad (2.2b) \quad \bar{\rho} \bar{\mathbf{v}}_t = \nabla_{\bar{\mathbf{x}}} \cdot \bar{\boldsymbol{\sigma}}, \quad \bar{\mathbf{x}} \in \bar{\Omega}_0,$$

$$95 \quad (2.2c) \quad \bar{\boldsymbol{\sigma}}_t = \bar{\lambda}(\nabla_{\bar{\mathbf{x}}} \cdot \bar{\mathbf{v}})\mathbf{I} + \bar{\mu}(\nabla_{\bar{\mathbf{x}}} \bar{\mathbf{v}} + (\nabla_{\bar{\mathbf{x}}} \bar{\mathbf{v}})^T), \quad \bar{\mathbf{x}} \in \bar{\Omega}_0,$$

97 where  $\bar{\mathbf{v}}(\bar{\mathbf{x}}, t)$  is the velocity of the solid, and  $\bar{\rho}$  is its density (assumed constant).

98 The fluid and solid are coupled at an interface described by  $\mathbf{x} \in \Gamma(t)$  in physical  
 99 space and  $\bar{\mathbf{x}} \in \bar{\Gamma}_0$  in the corresponding reference space. Along the interface, which is  
 100 assumed smooth, the following matching conditions hold:

$$101 \quad (2.3) \quad \mathbf{v} = \bar{\mathbf{v}}, \quad \boldsymbol{\sigma} \mathbf{n} = \bar{\boldsymbol{\sigma}} \mathbf{n}, \quad \mathbf{x} \in \Gamma(t),$$

102 where  $\mathbf{n}(\mathbf{x}, t)$  is the outward unit normal to the fluid domain. Suitable boundary  
 103 conditions are applied on the boundaries of the fluid and solid domains not included  
 104 in  $\Gamma(t)$ , and initial conditions on  $\mathbf{v}$ ,  $\bar{\mathbf{u}}$  and  $\bar{\mathbf{v}}$  are set to close the problem.

105 **3. AMP interface conditions and algorithm.** In this section, we derive the  
 106 AMP interface conditions at a continuous level and discuss their implementation in  
 107 the AMP algorithm. The derivation follows the work in [6], but there are important  
 108 modifications required to accommodate the IMEX fractional-step scheme used in the  
 109 AMP algorithm to efficiently advance the fluid. These modifications are guided by a  
 110 consideration of the behavior of the AMP interface conditions in the limits of very  
 111 light and very heavy solids.

112 **3.1. AMP interface conditions.** The starting point for the derivation is the  
 113 matching conditions involving velocity and stress in (2.3). Following [6], a linear com-  
 114 bination of these conditions are expressed in terms of Riemann variables corresponding  
 115 to the the outgoing solid characteristics, i.e.

$$116 \quad (3.1a) \quad -p + \mathbf{n}^T \boldsymbol{\tau} \mathbf{n} + \bar{z}_p \mathbf{n}^T \mathbf{v} = \mathbf{n}^T \bar{\boldsymbol{\sigma}} \mathbf{n} + \bar{z}_p \mathbf{n}^T \bar{\mathbf{v}}, \quad \mathbf{x} \in \Gamma(t),$$

$$117 \quad (3.1b) \quad \mathbf{t}_m^T \boldsymbol{\tau} \mathbf{n} + \bar{z}_s \mathbf{t}_m^T \mathbf{v} = \mathbf{t}_m^T \bar{\boldsymbol{\sigma}} \mathbf{n} + \bar{z}_s \mathbf{t}_m^T \bar{\mathbf{v}}, \quad m = 1, 2, \quad \mathbf{x} \in \Gamma(t),$$

119 where  $\mathbf{n}$  is the unit normal,  $\mathbf{t}_m$ ,  $m = 1, 2$ , are mutually orthogonal unit vectors tangent  
 120 to the interface. The Riemann variables in the solid are given by the right-hand side

121 of (3.1), and can be obtained from the first-order equations in (2.2b)–(2.2c) projected  
 122 onto the normal to the interface. The impedances,  $\bar{z}_p = \bar{\rho}\bar{c}_p$  and  $\bar{z}_s = \bar{\rho}\bar{c}_s$ , involve  
 123 the characteristic velocities of the solid given by  $\bar{c}_p = \sqrt{(\lambda + 2\bar{\mu})/\bar{\rho}}$  and  $\bar{c}_s = \sqrt{\bar{\mu}/\bar{\rho}}$ .  
 124 In the AMP algorithm, the conditions in (3.1) are interpreted as providing interface  
 125 conditions for the fluid in terms of the outgoing characteristic quantities of the solid,  
 126 assumed known from a previous stage of the algorithm. While these conditions, along  
 127 with  $\nabla \cdot \mathbf{v} = 0$  for  $\mathbf{x} \in \Gamma(t)$ , are sufficient conditions for the fluid equations in velocity-  
 128 pressure form, a further manipulation is required to obtain suitable conditions to  
 129 be used for the fractional-step solver. The objective is to separate the conditions  
 130 in (3.1) to obtain conditions to be used in the IMEX time-stepping scheme for the  
 131 fluid velocity and pressure.

132 For the Poisson problem for the fluid pressure, the interface condition in (3.1a) is  
 133 used with the momentum equation in (2.1a) to derive a Robin condition for the pres-  
 134 sure that balances accelerations. The momentum equation involves the acceleration  
 135 of the fluid, and this quantity may be obtained on the moving fluid-solid interface  
 136 using the Taylor approximation

$$137 \quad (3.2) \quad \mathbf{v}(\mathbf{x}, t - \Delta t) \Big|_{\mathbf{x}=\mathcal{P}(t-\Delta t)} \approx (\mathbf{v}(\mathbf{x}, t) - \Delta t D_t \mathbf{v}(\mathbf{x}, t)) \Big|_{\mathbf{x}=\mathcal{P}(t)},$$

139 where  $D_t = \partial_t + \mathbf{v} \cdot \nabla$  is the material derivative,  $\mathcal{P}(t)$  is a point on the moving interface  
 140 and  $\Delta t$  is a time-step. The corresponding approximation for the solid is

$$141 \quad (3.3) \quad \bar{\mathbf{v}}(\bar{\mathbf{x}}, t - \Delta t) \Big|_{\bar{\mathbf{x}}=\bar{\mathcal{P}}_0} \approx (\bar{\mathbf{v}}(\bar{\mathbf{x}}, t) - \Delta t \bar{\mathbf{v}}_t(\bar{\mathbf{x}}, t)) \Big|_{\bar{\mathbf{x}}=\bar{\mathcal{P}}_0},$$

143 where  $\bar{\mathcal{P}}_0$  is the Lagrangian position associated with  $\mathcal{P}(t)$ . Using (3.2) and (3.3)  
 144 in (3.1a), and assuming the fluid and solid velocities match on the interface at times  
 145  $t - \Delta t$  and  $t$ , we obtain

$$146 \quad -p + \mathbf{n}^T \boldsymbol{\tau} \mathbf{n} + \bar{z}_p \Delta t \mathbf{n}^T D_t \mathbf{v} = \mathbf{n}^T \bar{\boldsymbol{\sigma}} \mathbf{n} + \bar{z}_p \Delta t \mathbf{n}^T \bar{\mathbf{v}}_t, \quad \mathbf{x} \in \Gamma(t).$$

148 We may now eliminate the fluid acceleration using (2.1a) to obtain the following Robin  
 149 condition for the fluid pressure:

$$150 \quad (3.4) \quad -p - \frac{\bar{z}_p \Delta t}{\rho} \partial_n p = \mathbf{n}^T (\bar{\boldsymbol{\sigma}} \mathbf{n} - \boldsymbol{\tau} \mathbf{n}) + \bar{z}_p \Delta t \mathbf{n}^T (\bar{\mathbf{v}}_t + \nu \nabla \times \nabla \times \mathbf{v}), \quad \mathbf{x} \in \Gamma(t),$$

152 where  $\partial_n = \mathbf{n} \cdot \nabla$  is the normal derivative and  $\nu = \mu/\rho$  is the kinematic viscosity  
 153 of the fluid. Following [17], we have used the identity,  $\Delta \mathbf{v} = -\nabla \times \nabla \times \mathbf{v}$ , noting  
 154 that  $\nabla \cdot \mathbf{v} = 0$ , to replace  $\Delta \mathbf{v}$  on the right-hand side of (3.4) in favor of the curl-  
 155 curl operator. This is done for improved stability of the fractional-step scheme. The  
 156 condition in (3.4), along with suitable conditions for  $\mathbf{x} \in \partial\Omega(t) \setminus \Gamma(t)$ , is used for the  
 157 Poisson equation in (2.1b) for the pressure.

158 As was noted in [6], the remaining interface conditions in (3.1b), together with  
 159 the continuity equation, can be used as boundary conditions to advance the fluid  
 160 velocity. This was found to be an effective approach for an explicit integration of the  
 161 momentum equations. To ensure that the fluid velocity and tractions match at the  
 162 end of the time step, an interface projection is performed to give a common interface  
 163 velocity  $\mathbf{v}^I$  and interface traction  $\boldsymbol{\sigma}^I \mathbf{n}$ . In analogy to the interface projection used  
 164 for compressible fluids in [11, 5, 4], which is based on a characteristic analysis, the

165 projection for incompressible fluids is also proposed to be of the form of an impedance-  
 166 weighted average. A compact definition of this projection operator is given by

$$\begin{aligned}
 167 \quad \mathbf{w}^I &= \mathcal{Z}_I(z_f, \mathbf{v}, \boldsymbol{\sigma}, \bar{z}_p, \bar{z}_s, \bar{\mathbf{v}}, \bar{\boldsymbol{\sigma}}) = (\mathbf{n}^T \mathbf{w}^I) \mathbf{n} + \sum_{m=1}^2 (\mathbf{t}_m^T \mathbf{w}^I) \mathbf{t}_m, \\
 168 \quad \mathbf{n}^T \mathbf{w}^I &\stackrel{\text{def}}{=} \frac{1}{z_f + \bar{z}_p} \{z_f \mathbf{n}^T \mathbf{v} + \bar{z}_p \mathbf{n}^T \bar{\mathbf{v}} + \mathbf{n}^T (\bar{\boldsymbol{\sigma}} \mathbf{n} - \boldsymbol{\sigma} \mathbf{n})\}, \\
 169 \quad \mathbf{t}_m^T \mathbf{w}^I &\stackrel{\text{def}}{=} \frac{1}{z_f + \bar{z}_s} \{z_f \mathbf{t}_m^T \mathbf{v} + \bar{z}_s \mathbf{t}_m^T \bar{\mathbf{v}} + \mathbf{t}_m^T (\bar{\boldsymbol{\sigma}} \mathbf{n} - \boldsymbol{\sigma} \mathbf{n})\}, \quad m = 1, 2. \\
 170
 \end{aligned}$$

171 In terms of  $\mathcal{Z}_I$ , the velocity and stress projections are

$$\begin{aligned}
 172 \quad (3.6a) \quad \mathbf{v}^I &= \mathcal{Z}_I(z_f, \mathbf{v}, \boldsymbol{\sigma}, \bar{z}_p, \bar{z}_s, \bar{\mathbf{v}}, \bar{\boldsymbol{\sigma}}), \\
 173 \quad (3.6b) \quad \boldsymbol{\sigma}^I \mathbf{n} &= \mathcal{Z}_I(z_f^{-1}, \boldsymbol{\sigma}, \mathbf{v}, \bar{z}_p^{-1}, \bar{z}_s^{-1}, \bar{\boldsymbol{\sigma}}, \bar{\mathbf{v}}),
 \end{aligned}$$

175 noting the order of arguments for  $\boldsymbol{\sigma}^I \mathbf{n}$ . These projections introduce a *fluid impedance*,  
 176  $z_f$ , which is well defined for compressible fluids, but has no obvious definition for  
 177 incompressible fluids. However, a local analysis of a semi-discrete approximation to  
 178 the governing equations given in Section 4 suggests a form for  $z_f$  given by

$$179 \quad (3.7) \quad z_f \stackrel{\text{def}}{=} \mathcal{C}_{\text{AM}} \left( \frac{\rho h}{\Delta t} \right) + \mathcal{C}_{\text{AD}} \left( \frac{\mu}{h} \right),$$

180 where  $h$  is an appropriate mesh spacing and  $(\mathcal{C}_{\text{AM}}, \mathcal{C}_{\text{AD}})$  are constants whose approx-  
 181 imate values are provided by the analysis.

182 For the IMEX scheme considered here, a further modification of the previous  
 183 approach in [6] is required in the implementation of the interface conditions for the  
 184 fluid velocity. The issue is informed by considering the limits of very light and heavy  
 185 solids. In the limit of a very light solid ( $\bar{z}_p, \bar{z}_s \rightarrow 0$ ), for example, the Robin condition  
 186 in (3.4) becomes a Dirichlet condition for the pressure, while the interface conditions  
 187 in (3.1b) reduce to matching conditions involving the shear stress of the fluid. The  
 188 latter conditions, along with the continuity constraint, provide Neumann conditions  
 189 on the fluid velocity. These conditions for the fluid pressure and velocity correspond  
 190 to those for a free surface, and the latter are suitable for the implicit solution of the  
 191 fluid velocity in the IMEX fractional-step scheme.

192 The difficulty is revealed in the limit of a very heavy solid ( $\bar{z}_p, \bar{z}_s \rightarrow \infty$ ). In  
 193 this limit, the Robin condition in (3.4) becomes a Neumann condition for the fluid  
 194 pressure balancing the acceleration of the interface as determined by the solid. This  
 195 condition is analogous to the usual Neumann boundary condition for the pressure  
 196 at a rigid boundary obtained from the fluid momentum equations as a compatibility  
 197 condition (see [17] for example). The interface conditions in (3.1b) reduce to matching  
 198 conditions involving the tangential components of velocity. However, the matching  
 199 condition on the normal component of velocity,  $\mathbf{n}^T \mathbf{v} = \mathbf{n}^T \bar{\mathbf{v}}$ , implied by (3.1a) in  
 200 the limit of a heavy solid has been lost in the derivation of (3.4). A remedy can be  
 201 obtained by using the interface projection for the normal component of the velocity  
 202 in (3.6) as a boundary condition for the implicit solution of the fluid velocity in the  
 203 IMEX fractional-step scheme. The implementation of this approach is described next  
 204 in the discussion of the AMP algorithm.

205 **3.2. AMP algorithm.** Algorithm 3.1 provides a concise description of the AMP  
 206 time-stepping scheme (see [24] for additional details of the implementation of the

207 algorithm). The algorithm advances the solution from a time  $t^n$  to  $t^{n+1} = t^n + \Delta t$ . It  
 208 is assumed that the fluid domain is represented by a grid consisting of interior points  
 209  $\mathbf{i} \in \Omega_h$ , boundary points  $\mathbf{i} \in \partial\Omega_h$  and interface points  $\mathbf{i} \in \Gamma_h$ , where  $\mathbf{i} = (i_1, i_2, i_3)$   
 210 is a multi-index. Similarly, the solid reference domain is covered by a grid with  
 211 interior points  $\bar{\mathbf{i}} \in \bar{\Omega}_h$ , boundary points  $\bar{\mathbf{i}} \in \partial\bar{\Omega}_h$  and interface points  $\bar{\mathbf{i}} \in \bar{\Gamma}_h$ . Discrete  
 212 operators, such as  $\nabla_h$  and  $\Delta_h$ , denote approximations of the corresponding differential  
 213 operators on the grid.

---

**Algorithm 3.1** Added-mass partitioned (AMP) scheme

---

// Predictor steps

1. Predict solid:

$$\begin{cases} \bar{\mathbf{u}}_{\bar{\mathbf{i}}}^{(p)} = \bar{\mathbf{u}}_{\bar{\mathbf{i}}}^n + \Delta t \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^n + \frac{\Delta t^2}{2\rho} \bar{\nabla}_h \cdot \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^n, & \bar{\mathbf{i}} \in \bar{\Omega}_h, \\ \bar{\mathbf{q}}_{\bar{\mathbf{i}}}^{(p)} = \bar{\mathbf{q}}_{\bar{\mathbf{i}}}^n - \Delta t \sum_{m=1}^3 \frac{1}{\Delta \bar{x}_m} (\bar{\mathbf{F}}_{m, \bar{\mathbf{i}}}^+ - \bar{\mathbf{F}}_{m, \bar{\mathbf{i}}}^-), & \bar{\mathbf{i}} \in \Omega_h, \end{cases}$$

2. Predict fluid grid: advance fluid grid to  $t^{n+1}$  using  $\bar{\mathbf{u}}_{\bar{\mathbf{i}}}^{(p)}$  for  $\bar{\mathbf{i}} \in \bar{\Gamma}_h$ , and compute grid velocity.

3. Predict fluid velocity:

$$\begin{cases} \mathbf{v}_{\mathbf{i}}^{(p)} = \mathbf{v}_{\mathbf{i}}^n + \frac{\Delta t}{2} (3\mathbf{N}_h(\mathbf{v}_{\mathbf{i}}^n, p_{\mathbf{i}}^n) - \mathbf{N}_h(\mathbf{v}_{\mathbf{i}}^{n-1}, p_{\mathbf{i}}^{n-1})) + \frac{\Delta t}{2} (\mathbf{L}_h(\mathbf{v}_{\mathbf{i}}^{(p)}) + \mathbf{L}_h(\mathbf{v}_{\mathbf{i}}^n)), & \mathbf{i} \in \Omega_h \setminus \Gamma_h, \\ \mathbf{t}_m^T \boldsymbol{\tau}_{\mathbf{i}}^{(p)} \mathbf{n} + \bar{z}_s \mathbf{t}_m^T \mathbf{v}_{\mathbf{i}}^{(p)} = \mathbf{t}_m^T \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^{(p)} \mathbf{n} + \bar{z}_s \mathbf{t}_m^T \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^{(p)}, & \mathbf{i} \in \Gamma_h, \bar{\mathbf{i}} \in \bar{\Gamma}_h, \\ \nabla_h \cdot \mathbf{v}_{\mathbf{i}}^{(p)} = 0, & \mathbf{i} \in \Gamma_h, \\ \mathbf{n}^T \mathbf{v}_{\mathbf{i}}^{(p)} = \frac{z_f}{z_f + \bar{z}_p} \mathbf{n}^T \mathbf{V}_h(\mathbf{v}_{\mathbf{i}}^{(p)}) + \frac{\bar{z}_p}{z_f + \bar{z}_p} \mathbf{n}^T \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^{(p)}, \quad \mathbf{t}_m^T \mathbf{v}_{\mathbf{i}}^{(p)} = \mathbf{t}_m^T \mathbf{V}_h(\mathbf{v}_{\mathbf{i}}^{(p)}), & \mathbf{i} \in \Gamma_h, \bar{\mathbf{i}} \in \bar{\Gamma}_h, \\ \text{Velocity boundary conditions on } \partial\Omega_h \setminus \Gamma_h. \end{cases}$$

4. Predict fluid pressure:

$$\begin{cases} \Delta_h p_{\mathbf{i}}^{(p)} = -\rho \nabla_h \mathbf{v}_{\mathbf{i}}^{(p)} : (\nabla_h \mathbf{v}_{\mathbf{i}}^{(p)})^T + \alpha_i \nabla_h \cdot \mathbf{v}_{\mathbf{i}}^{(p)}, & \mathbf{i} \in \Omega_h, \\ -p_{\mathbf{i}}^{(p)} - \frac{\bar{z}_p \Delta t}{\rho} (\mathbf{n} \cdot \nabla_h) p_{\mathbf{i}}^{(p)} = \mathbf{n}^T (\bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^{(p)} \mathbf{n} - \boldsymbol{\tau}_{\mathbf{i}}^{(p)} \mathbf{n}) \\ \quad + \bar{z}_p \Delta t \mathbf{n}^T ((\bar{\mathbf{v}}_{\bar{\mathbf{i}}})_{\bar{\mathbf{i}}}^{(p)} + \nu \nabla_h \times \nabla_h \times \mathbf{v}_{\mathbf{i}}^{(p)}), & \mathbf{i} \in \Gamma_h, \bar{\mathbf{i}} \in \bar{\Gamma}_h, \\ \text{Pressure boundary conditions on } \partial\Omega_h \setminus \Gamma_h. \end{cases}$$

5. Project solid interface for  $\bar{\mathbf{i}} \in \bar{\Gamma}_h, \mathbf{i} \in \Gamma_h$ .

$$\begin{cases} \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^I = \mathcal{Z}_I(z_f, \mathbf{v}_{\mathbf{i}}^{(p)}, \boldsymbol{\sigma}_{\mathbf{i}}^{(p)}, \bar{z}_p, \bar{z}_s, \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^{(p)}, \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^{(p)}) \\ \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^I \mathbf{n} = \mathcal{Z}_I(z_f^{-1}, \boldsymbol{\sigma}_{\mathbf{i}}^{(p)}, \mathbf{v}_{\mathbf{i}}^{(p)}, \bar{z}_p^{-1}, \bar{z}_s^{-1}, \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^{(p)}, \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^{(p)}) \\ \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^{(p)} \leftarrow \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^I, \quad \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^{(p)} \mathbf{n} \leftarrow \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^I \mathbf{n}, \end{cases}$$

Apply solid boundary conditions and set all ghost points.

// Corrector steps

6. Correct fluid grid: recompute grid velocity using  $\bar{\mathbf{v}}_{\bar{\mathbf{i}}}^I$  for  $\bar{\mathbf{i}} \in \bar{\Gamma}_h$ .

7. Correct fluid velocity:

$$\begin{cases} \mathbf{v}_{\mathbf{i}}^{n+1} = \mathbf{v}_{\mathbf{i}}^n + \frac{\Delta t}{2} (\mathbf{N}_h(\mathbf{v}_{\mathbf{i}}^{(p)}, p_{\mathbf{i}}^{(p)}) + \mathbf{N}_h(\mathbf{v}_{\mathbf{i}}^n, p_{\mathbf{i}}^n)) + \frac{\Delta t}{2} (\mathbf{L}_h(\mathbf{v}_{\mathbf{i}}^{n+1}) + \mathbf{L}_h(\mathbf{v}_{\mathbf{i}}^n)), & \mathbf{i} \in \Omega_h \setminus \Gamma_h, \\ \mathbf{t}_m^T \boldsymbol{\tau}_{\mathbf{i}}^{n+1} \mathbf{n} + \bar{z}_s \mathbf{t}_m^T \mathbf{v}_{\mathbf{i}}^{n+1} = \mathbf{t}_m^T \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^I \mathbf{n} + \bar{z}_s \mathbf{t}_m^T \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^I, & \mathbf{i} \in \Gamma_h, \bar{\mathbf{i}} \in \bar{\Gamma}_h, \\ \nabla_h \cdot \mathbf{v}_{\mathbf{i}}^{n+1} = 0, & \mathbf{i} \in \Gamma_h, \\ \mathbf{n}^T \mathbf{v}_{\mathbf{i}}^{n+1} = \frac{z_f}{z_f + \bar{z}_p} \mathbf{n}^T \mathbf{V}_h(\mathbf{v}_{\mathbf{i}}^{n+1}) + \frac{\bar{z}_p}{z_f + \bar{z}_p} \mathbf{n}^T \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^I, \quad \mathbf{t}_m^T \mathbf{v}_{\mathbf{i}}^{n+1} = \mathbf{t}_m^T \mathbf{V}_h(\mathbf{v}_{\mathbf{i}}^{n+1}), & \mathbf{i} \in \Gamma_h, \bar{\mathbf{i}} \in \bar{\Gamma}_h, \\ \text{Velocity boundary conditions on } \partial\Omega_h \setminus \Gamma_h. \end{cases}$$

8. Correct fluid pressure.

$$\begin{cases} \Delta_h p_{\mathbf{i}}^{n+1} = -\rho \nabla_h \mathbf{v}_{\mathbf{i}}^{n+1} : (\nabla_h \mathbf{v}_{\mathbf{i}}^{n+1})^T + \alpha_i \nabla_h \cdot \mathbf{v}_{\mathbf{i}}^{n+1}, & \mathbf{i} \in \Omega_h, \\ -p_{\mathbf{i}}^{n+1} - \frac{\bar{z}_p \Delta t}{\rho} (\mathbf{n} \cdot \nabla_h) p_{\mathbf{i}}^{n+1} = \mathbf{n}^T (\bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^I \mathbf{n} - \boldsymbol{\tau}_{\mathbf{i}}^{n+1} \mathbf{n}) \\ \quad + \bar{z}_p \Delta t \mathbf{n}^T ((\bar{\mathbf{v}}_{\bar{\mathbf{i}}})_{\bar{\mathbf{i}}}^I + \nu \nabla_h \times \nabla_h \times \mathbf{v}_{\mathbf{i}}^{n+1}), & \mathbf{i} \in \Gamma_h, \bar{\mathbf{i}} \in \bar{\Gamma}_h, \\ \text{Pressure boundary conditions on } \partial\Omega_h \setminus \Gamma_h. \end{cases}$$

9. Correct solid interface.

$$\begin{cases} \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^{n+1} = \mathbf{v}_{\mathbf{i}}^{n+1}, & \bar{\mathbf{i}} \in \bar{\Gamma}_h, \mathbf{i} \in \Gamma_h, \\ \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}}^{n+1} \mathbf{n} = \boldsymbol{\sigma}_{\mathbf{i}}^{n+1} \mathbf{n}, & \bar{\mathbf{i}} \in \bar{\Gamma}_h, \mathbf{i} \in \Gamma_h, \\ \text{Reset ghost points corresponding to } \bar{\mathbf{i}} \in \bar{\Gamma}_h. \end{cases}$$


---

214 The time-stepping scheme uses a predictor-corrector approach. Steps 1–5 of Al-  
 215 gorithm 3.1 describe the predictor steps. Predicted values for the solid displacement



216  $\bar{\mathbf{u}}_{\bar{\mathbf{i}}}$  are obtained in Step 1 using a Lax-Wendroff-type scheme for (2.2a), while the solid  
 217 velocity and stress  $\bar{\mathbf{q}}_{\bar{\mathbf{i}}} = (\bar{\mathbf{v}}_{\bar{\mathbf{i}}}, \bar{\boldsymbol{\sigma}}_{\bar{\mathbf{i}}})$  are advanced using a Godunov-type scheme for (2.2b)  
 218 and (2.2c) with numerical fluxes  $\bar{\mathbf{F}}_{m, \bar{\mathbf{i}}}^{\pm}$  corresponding to the  $\bar{x}_m$ -direction. In Step 2,  
 219 the solid displacement is used to compute the deformed fluid grid at time  $t^{n+1}$ .

220 The fluid velocity is predicted in Step 3. Here,  $\mathbf{N}_h$  and  $\mathbf{L}_h$  represent grid operators  
 221 associated with the explicit and implicit terms in the velocity update, respectively,  
 222 given by

$$223 \quad \mathbf{N}_h(\mathbf{v}_i, p_i) \stackrel{\text{def}}{=} -((\mathbf{v}_i - \dot{\mathbf{x}}_i) \cdot \nabla_h) \mathbf{v}_i - \frac{1}{\rho} \nabla_h p_i, \quad \mathbf{L}_h(\mathbf{v}_i) \stackrel{\text{def}}{=} \nu \Delta_h \mathbf{v}_i,$$

225 where  $\dot{\mathbf{x}}_i$  is the velocity of the grid. The explicit terms are advanced using an Adams-  
 226 Bashforth scheme, while the implicit terms use Crank-Nicholson. The boundary condi-  
 227 tions on the interface makes use of a predicted velocity, coming from the interior  
 228 equation applied on the boundary, and defined by

$$229 \quad \mathbf{V}_h^p(\mathbf{v}_i^{(p)}) \stackrel{\text{def}}{=} \mathbf{v}_i^n + \frac{\Delta t}{2} (3\mathbf{N}_h(\mathbf{v}_i^n, p_i^n) - \mathbf{N}_h(\mathbf{v}_i^{n-1}, p_i^{n-1})) + \frac{\Delta t}{2} (\mathbf{L}_h(\mathbf{v}_i^{(p)}) + \mathbf{L}_h(\mathbf{v}_i^n)).$$

231 In particular, this velocity is used in the impedance-weighted average condition

$$232 \quad (3.8) \quad \mathbf{n}^T \mathbf{v}_i^{(p)} = \frac{z_f}{z_f + \bar{z}_p} \mathbf{n}^T \mathbf{V}_h^p(\mathbf{v}_i^{(p)}) + \frac{\bar{z}_p}{z_f + \bar{z}_p} \mathbf{n}^T \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^{(p)}, \quad \mathbf{i} \in \Gamma_h, \quad \bar{\mathbf{i}} \in \bar{\Gamma}_h,$$

234 which is obtained from the projection in (3.6). Here the the term involving the jump  
 235 in the stress is dropped (as it is apparently not essential to the scheme and simplifies  
 236 the implementation, see also [11]). Notice that (3.8) is an implicit condition on  $\mathbf{v}_i^{(p)}$   
 237 which appears on the left- and right-hand sides. In the light-solid limit ( $\bar{z}_p \rightarrow 0$ ),  
 238 the boundary condition in (3.8) reduces to  $\mathbf{n}^T \mathbf{v}_i^{(p)} = \mathbf{n}^T \mathbf{V}_h^p(\mathbf{v}_i^{(p)})$ , which simply sets  
 239 the normal component of the fluid velocity to be equal to that given by the interior  
 240 time-stepping scheme applied on the boundary. In the heavy-solid limit ( $\bar{z}_p \rightarrow \infty$ ),  
 241 (3.8) becomes  $\mathbf{n}^T \mathbf{v}_i^{(p)} = \mathbf{n}^T \bar{\mathbf{v}}_{\bar{\mathbf{i}}}^{(p)}$ , which recovers the desired matching condition. Our  
 242 later analysis of a viscous model problem (Section 5) and subsequent numerical results  
 243 (Section 6), verify that the boundary conditions used to advance the fluid velocity in  
 244 the fractional-step scheme lead to stable and accurate results for all solid densities we  
 245 have considered.

246 Steps 4 and 5 complete the set of steps belonging to the predictor stage of the  
 247 algorithm. The predicted fluid pressure is computed in Step 4 by solving a discrete  
 248 Poisson problem. This elliptic problem uses a discrete approximation of the Robin  
 249 condition in (3.4). Finally, interface values for the solid velocity and traction are  
 250 obtained in Step 5 using the impedance-weighted projections in (3.6). These interface  
 251 values overwrite the corresponding predicted values of the solid on the boundary.

252 The set of corrector steps consisting of Steps 6–9 essentially mirror those of the  
 253 predictor. In an important final Step 9, the solid velocity and traction are set equal  
 254 to the corrected fluid values.

255 **4. Derivation of the fluid impedance.** The focus of this section is an analysis  
 256 of an FSI problem that guides the choice for the fluid impedance  $z_f$  introduced in (3.7)  
 257 and required in the interface projections (3.6). Previously in [6], the choice of fluid  
 258 impedance was found to be somewhat arbitrary and a choice was made of  $z_f = \rho H / \Delta t$ ,  
 259 where  $H$  was a measure of the depth of the fluid layer. With the current IMEX scheme,  
 260 the viscous CFL number,  $\Lambda = \nu \Delta t / h^2$ , can be large in which case a new choice for

261  $z_f$  is needed to keep the scheme stable; a carefully chosen model problem is used for  
 262 this purpose.

263 Consider an FSI model problem in which the fluid occupies the two-dimensional  
 264 domain,  $\Omega$ , given by  $0 < x < L$ ,  $y > 0$ , while the solid exists on the domain,  $\bar{\Omega}_0$ , for  
 265  $0 < x < L$ ,  $y < 0$ . The fluid-solid interface,  $\Gamma$ , of length  $L$  is linearized about a flat  
 266 surface,  $y = 0$ . The equations governing the model problem are

$$267 \quad (4.1a) \quad \text{Fluid:} \quad \begin{cases} \rho \partial_t \mathbf{v} + \nabla p = \mu \Delta \mathbf{v}, & \mathbf{x} \in \Omega, \\ \Delta p = 0, & \mathbf{x} \in \Omega, \\ \nabla \cdot \mathbf{v} = 0, & \mathbf{x} \in \Gamma, \end{cases}$$

$$268 \quad (4.1b) \quad \text{Solid:} \quad \begin{cases} \bar{\rho} \partial_t \bar{\mathbf{v}} = \nabla \cdot \bar{\boldsymbol{\sigma}}, & \mathbf{x} \in \bar{\Omega}_0, \\ \partial_t \bar{\boldsymbol{\sigma}} = \bar{\lambda} (\nabla \cdot \mathbf{v}) \mathbf{I} + \bar{\mu} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T), & \mathbf{x} \in \bar{\Omega}_0, \end{cases}$$

$$269 \quad (4.1c) \quad \text{Interface:} \quad \begin{cases} \mathbf{v} = \bar{\mathbf{v}}, & \mathbf{x} \in \Gamma, \\ \boldsymbol{\sigma} \mathbf{n} = \bar{\boldsymbol{\sigma}} \mathbf{n}, & \mathbf{x} \in \Gamma. \end{cases}$$

270

271 Solutions of the model problem are assumed to be periodic in  $x$  with period equal to  $L$ ,  
 272 and bounded as  $y \rightarrow \pm\infty$ . The equations governing the fluid and solid are discretized  
 273 in the  $x$ -direction on a uniform grid,  $x_\ell = \ell \Delta x$  for  $\ell = 0, 1, \dots, N_x$ , with grid spacing  
 274  $\Delta x = L/N_x$ . Since the problem is periodic, each variable can be represented as a  
 275 discrete Fourier series

$$276 \quad (4.2) \quad q(x, y, t) = \sum_{k=-N_x/2}^{N_x/2} e^{2\pi i k x / L} \hat{q}_k(y, t), \quad x \in [0, L],$$

277

278 where  $\hat{q}_k(y, t)$  are the Fourier coefficient functions and  $N_x$  is an integer, assumed to be  
 279 even for convenience. Taking a finite Fourier transform of the fluid equations in (4.1a)  
 280 gives

$$281 \quad (4.3a) \quad \rho \partial_t v_1 + i k_x p = \mu (\partial_y^2 - k_x^2) v_1, \quad y > 0,$$

$$282 \quad (4.3b) \quad \rho \partial_t v_2 + \partial_y p = \mu (\partial_y^2 - k_x^2) v_2, \quad y > 0,$$

$$283 \quad (4.3c) \quad (\partial_y^2 - k_x^2) p = 0, \quad y > 0,$$

285 where  $k_x = 2\pi k / L$ . The hats and  $k$  subscripts on the coefficient functions in (4.3) have  
 286 been dropped for notational convenience. The equations for the Fourier coefficient  
 287 functions are now discretized in time. Define the grid functions  $\mathbf{v}^n(y) \approx \mathbf{v}(y, t^n)$   
 288 and  $p^n \approx p(y, t^n)$ , where  $t^n = n \Delta t$  for a (fixed) time step  $\Delta t$ . An implicit scheme  
 289 to advance the solution from  $t^n$  to  $t^{n+1}$ , based on backward-Euler time-stepping, is  
 290 given by

$$291 \quad (4.4a) \quad \rho \frac{v_1^{n+1} - v_1^n}{\Delta t} + i k_x p^{n+1} = \mu (\partial_y^2 - k_x^2) v_1^{n+1}, \quad y > 0,$$

$$292 \quad (4.4b) \quad \rho \frac{v_2^{n+1} - v_2^n}{\Delta t} + \partial_y p^{n+1} = \mu (\partial_y^2 - k_x^2) v_2^{n+1}, \quad y > 0,$$

$$293 \quad (4.4c) \quad (\partial_y^2 - k_x^2) p^{n+1} = 0. \quad y > 0.$$

295 Assume that the coefficient functions for the solid variables have been advanced to  $t =$   
 296  $t^{n+1}$  using an upwind scheme, for example, and that  $b_p^{n+1}$  and  $b_s^{n+1}$  are, respectively,  
 297 the normal and tangential components of the outgoing characteristic variables of the



298 solid at  $t^{n+1}$ . Using (3.1), the boundary conditions for the fluid at  $y = 0$  take the  
 299 form

$$300 \quad (4.5a) \quad -p^{n+1} + \tau_{22}^{n+1} - \bar{z}_p v_2^{n+1} = b_p^{n+1}, \quad y = 0,$$

$$301 \quad (4.5b) \quad \tau_{12}^{n+1} - \bar{z}_s v_1^{n+1} = b_s^{n+1}, \quad y = 0,$$

303 where the components of the fluid shear stress in (4.5) are given by

$$304 \quad \tau_{12}^{n+1} = \mu (ik_x v_2^{n+1} + \partial_y v_1^{n+1}), \quad \tau_{22}^{n+1} = 2\mu \partial_y v_2^{n+1}.$$

306 The implicit scheme in (4.4) with boundary conditions in (4.5) at  $y = 0$  and bound-  
 307 edness as  $y \rightarrow \infty$  determine the grid functions for the fluid at  $t^{n+1}$  in terms the fluid  
 308 velocity at  $t^n$  and the outgoing solid data  $(b_p^{n+1}, b_s^{n+1})$ .

309 Consider perturbations in the grid functions of the fluid at  $t^{n+1}$  for  $y > 0$  subject  
 310 to perturbations in the interface data  $b_p^{n+1}$  and  $b_s^{n+1}$  at  $y = 0$ . The variational  
 311 equations corresponding to (4.4) are

$$312 \quad \frac{\rho}{\Delta t} \delta V_1 + ik_x \delta P = \mu (\partial_y^2 - k_x^2) \delta V_1, \quad y > 0,$$

$$313 \quad \frac{\rho}{\Delta t} \delta V_2 + \partial_y \delta P = \mu (\partial_y^2 - k_x^2) \delta V_2, \quad y > 0,$$

$$314 \quad (\partial_y^2 - k_x^2) \delta P = 0, \quad y > 0,$$

316 where  $(\delta V_1, \delta V_2, \delta P)$  are small perturbations corresponding to  $(v_1^{n+1}, v_2^{n+1}, p^{n+1})$ . So-  
 317 lutions to these equations that remain bounded as  $y \rightarrow \infty$  are

$$318 \quad \delta V_1(y) = -\frac{1}{ik_x} \partial_y \delta V_2(y),$$

$$319 \quad \delta V_2(y) = \delta V_0 e^{-\beta y} + \frac{|k_x| \Delta t \delta P_0}{\rho} (e^{-|k_x|y} - e^{-\beta y}),$$

$$320 \quad \delta P(y) = \delta P_0 e^{-|k_x|y},$$

322 where  $\delta V_0 = \delta V_2(0)$ ,  $\delta P_0 = \delta P(0)$  and  $\beta = \sqrt{k_x^2 + \rho/(\mu \Delta t)}$ . Substituting the solution  
 323 for the perturbations of the fluid variables into the variational equations corresponding  
 324 to (4.5) for the interface conditions leads to the linear system

$$325 \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \delta V_0 \\ \delta P_0 \end{bmatrix} = \begin{bmatrix} \delta B_p \\ \delta B_s \end{bmatrix},$$

327 where  $a_{11} = -\mu |k_x| (2\gamma + 1/Z_p)$ ,  $a_{12} = -1 + 2\Lambda(\gamma - 1)$ ,  $a_{21} = i\mu k_x (\gamma^2 + 1 + \gamma/Z_s)$   
 328 and  $a_{22} = -i \operatorname{sgn}(k_x) (1 + \Lambda(\gamma - 1)/Z_s)$ . Here,  $(\delta B_p, \delta B_s)$  are small perturbations  
 329 corresponding to  $(b_p^{n+1}, b_s^{n+1})$ . The coefficients  $a_{ij}$  in the linear system are defined in  
 330 terms of the dimensionless parameters

$$331 \quad \Lambda \stackrel{\text{def}}{=} \nu k_x^2 \Delta t, \quad \gamma \stackrel{\text{def}}{=} \beta/|k_x| = \sqrt{1 + 1/\Lambda}, \quad Z_\alpha \stackrel{\text{def}}{=} \mu |k_x| / \bar{z}_\alpha, \quad \alpha = p \text{ or } s.$$

332 The solution of the linear system

$$333 \quad (4.8) \quad \delta V_0 = \frac{a_{22} \delta B_p - a_{12} \delta B_s}{a_{11} a_{22} - a_{12} a_{21}}, \quad \delta P_0 = \frac{a_{11} \delta B_s - a_{21} \delta B_p}{a_{11} a_{22} - a_{12} a_{21}},$$

335 determines the variation in the interface values of the fluid velocity and pressure in  
 336 terms of the variations in the outgoing characteristic variables of the solid.

337 The AMP algorithm uses impedance-weighted averages to set values for the ve-  
 338 locity and pressure at the interface. For example, the normal component of veloc-  
 339 ity at the interface is given by (3.6a). In terms of the variational problem, (3.6a)  
 340 reduces to  $\delta V_0 = -\delta B_p/(z_f + \bar{z}_p)$ , assuming that the fluid velocity and stress  
 341 on the right-hand side are held fixed. In view of the solution in (4.8), we have  
 342  $(z_f + \bar{z}_p)^{-1} = -a_{22}/(a_{11}a_{22} - a_{12}a_{21})$ , which, after some manipulation, gives

$$343 \quad (4.9) \quad z_f = \mu|k_x|R, \quad R \stackrel{\text{def}}{=} 2\gamma + \frac{(\gamma + Z_s(\gamma^2 + 1))(1 - 2\Lambda(\gamma - 1))}{\Lambda(\gamma - 1)(1 + Z_s(\gamma + 1))}.$$

344 Of particular interest are the limiting cases when the viscous CFL number,  $\Lambda$ , is small  
 345 and large. A straightforward analysis of the dimensionless parameter  $R$  in (4.9) gives  
 346  $R \sim 1/\Lambda$  for  $\Lambda \ll 1$  and  $R \sim 2$  for  $\Lambda \gg 1$ . In view of these limits, consider an  
 347 approximation to  $R$  given by

$$348 \quad \tilde{R} \stackrel{\text{def}}{=} 1/\Lambda + 2.$$

349 It is found that  $1 \leq R/\tilde{R} \leq (\sqrt{2} + 1)/2 \approx 1.21$  for all  $(\Lambda, Z_s)$ , so that the fluid  
 350 impedance given by

$$351 \quad (4.10) \quad z_f = \mu|k_x|\tilde{R} = \mu|k_x|(1/\Lambda + 2) = \rho/(|k_x|\Delta t) + 2\mu|k_x|,$$

352 is a good approximation of the more complicated form given in (4.9). The model  
 353 problem analysis of Section 5 confirms that this choice leads to a stable scheme.

354 Formula (4.10) provides the generic form of the fluid impedance we use, but it  
 355 remains to make a choice for  $k_x$  so that the approximation can be used for a discrete  
 356 approximation in physical coordinates (as opposed to the Fourier transformed space).  
 357 Note that in a discrete approximation, the magnitude of the possible wave numbers  $k_x$   
 358 appearing in (4.10) are bounded by approximately  $1/h$ , where  $h$  is a measure of the  
 359 grid spacing in the tangential direction. For the present model problem with the  
 360 the pseudo-spectral approximation (4.2), for example, we have  $|k_x| \leq \pi/\Delta x$ , while  
 361 a second-order difference approximation would roughly imply  $|k_x| \leq 2/\Delta x$ . Expe-  
 362 rience [8] shows that added-damping instabilities are generally caused by relatively  
 363 high-frequency modes on the grid, and this suggests taking  $|k_x| = 1/h$  which leads to  
 364 a definition of the fluid impedance of the form

$$365 \quad z_f \stackrel{\text{def}}{=} \mathcal{C}_{\text{AM}}\left(\frac{\rho h}{\Delta t}\right) + \mathcal{C}_{\text{AD}}\left(\frac{\mu}{h}\right),$$

367 as was done in (3.7). The extensive numerical results in Section 6 and [24] confirm that  
 368 this is an appropriate choice, and furthermore that the scheme is rather insensitive to  
 369 the choice of  $h$ ,  $\mathcal{C}_{\text{AM}}$  and  $\mathcal{C}_{\text{AD}}$ .

370 **5. Stability analysis of an FSI model problem.** The stability of the AMP  
 371 algorithm is explored in the context of an FSI model problem involving a viscous  
 372 incompressible (Stokes) fluid in contact with a simplified elastic solid. This analysis  
 373 extends the work in [6] to the case of a viscous fluid where both added-mass and  
 374 added-damping effects are important, and for an IMEX-type scheme in the fluid.

375 We will compare the stability of the AMP scheme to that of the traditional  
 376 partitioned (TP) scheme and the anti-traditional partitioned (ATP) scheme. In the  
 377 TP scheme, the solid provides a Dirichlet (no-slip) boundary condition for the fluid,  
 378 and then the fluid supplies a Neumann (traction) boundary condition for the solid.  
 379 The ATP scheme reverses the role of the solid and fluid. In this scheme, the solid  
 380 provides a Neumann (traction) boundary condition for the fluid and the fluid supplies  
 381 a Dirichlet (no-slip) boundary condition for the solid.

382 **5.1. Model problem.** The viscous model problem analyzed here is similar to  
383 the one discussed in Section 4. An incompressible Stokes fluid satisfies the system of  
384 equations in (4.1a) for  $\mathbf{x} \in \Omega$ . The solid satisfies (4.1b) for  $\mathbf{x} \in \bar{\Omega}_0$ , but with  $\bar{\lambda}$  set  
385 equal to  $-\bar{\mu}$ . While this choice may not correspond to any actual solid, it is a useful  
386 choice mathematically as it simplifies the equations for the solid somewhat since the  
387 compressive wave speed becomes equal to the shear wave speed,  $\bar{c} = \sqrt{\bar{\mu}/\bar{\rho}}$ , which is  
388 particularly relevant for the viscous model problem. It is convenient to consider the  
389 hyperbolic equations for the solid in characteristic form. These equations are

$$390 \quad (5.1a) \quad \partial_t a_1 - \bar{c} \partial_y a_1 = \bar{c} (\partial_x d - \partial_x b_2), \quad \mathbf{x} \in \bar{\Omega}_0,$$

$$391 \quad (5.1b) \quad \partial_t b_1 + \bar{c} \partial_y b_1 = \bar{c} (\partial_x a_2 - \partial_x d), \quad \mathbf{x} \in \bar{\Omega}_0,$$

$$392 \quad (5.1c) \quad \partial_t a_2 - \bar{c} \partial_y a_2 = \bar{c} \partial_x b_1, \quad \mathbf{x} \in \bar{\Omega}_0,$$

$$393 \quad (5.1d) \quad \partial_t b_2 + \bar{c} \partial_y b_2 = -\bar{c} \partial_x a_1, \quad \mathbf{x} \in \bar{\Omega}_0,$$

$$394 \quad (5.1e) \quad \partial_t d = 0, \quad \mathbf{x} \in \bar{\Omega}_0,$$

396 where  $a_m = \bar{\sigma}_{m2} + \bar{z} \bar{v}_m$  and  $b_m = \bar{\sigma}_{m2} - \bar{z} \bar{v}_m$ , for  $m = 1, 2$  and  $\bar{z} = \bar{\rho} \bar{c}$ , are the  
397 variables associated with the incoming and outgoing characteristics at the interface,  
398 respectively, and  $d = \bar{\sigma}_{11} + \bar{\sigma}_{22}$ . The interface is linearized about a flat surface  $\Gamma$   
399 given by  $y = 0$ , and the matching conditions between the fluid and the solid are given  
400 in (4.1c).

401 **5.2. Discretization.** The discretization of the equations in the  $x$ -direction fol-  
402 lows the approach used previously in Section 4. The equations for the fluid are trans-  
403 formed using the finite Fourier series in (4.2), which results in a system of equations  
404 for the corresponding Fourier coefficient functions given in (4.1a). These equations  
405 are then discretized in time using an IMEX scheme given by

$$406 \quad (5.2a) \quad v_1^{n+1} = v_1^n - \frac{ik_x \Delta t}{\rho} p^n + \nu \Delta t (-k_x^2 + \partial_y^2) v_1^{n+1},$$

$$407 \quad (5.2b) \quad v_2^{n+1} = v_2^n - \frac{\Delta t}{\rho} \partial_y p^n + \nu \Delta t (-k_x^2 + \partial_y^2) v_2^{n+1},$$

$$408 \quad (5.2c) \quad (-k_x^2 + \partial_y^2) p^{n+1} = 0.$$

410 Here,  $v_1^n(y)$ ,  $v_2^n(y)$  and  $p^n(y)$  approximate  $v_1(y, t^n)$ ,  $v_2(y, t^n)$  and  $p(y, t^n)$ , respectively,  
411 at  $t^n = n\Delta t$  for a fixed time step  $\Delta t$ . Recall that  $k_x = 2\pi k/L$  and  $\nu = \mu/\rho$ , and that  
412 the hats and  $k$  subscripts on the Fourier coefficients have been suppressed. Note that  
413 the components of the fluid velocity are advanced in time using (implicit) backward  
414 Euler for the viscous terms and (explicit) forward Euler for the pressure gradient  
415 terms. An elliptic equation is solved at each time step to update the pressure. It is  
416 convenient to keep the discrete equations for the fluid variables continuous in  $y$ , and  
417 we assume that solutions are bounded as  $y \rightarrow \infty$ .

418 The characteristic equations for the solid in (5.1) are similarly transformed using  
419 the finite Fourier series in (4.2), and then the resulting equations are discretized in  
420 time and space using an upwind-type scheme given by

$$421 \quad (5.3a) \quad a_{1,j}^{n+1} = a_{1,j}^n + \bar{c} \Delta t (a_{1,j+1}^n - a_{1,j}^n) / \Delta y + i \bar{c} k_x \Delta t (d_j^{n+1} - b_{2,j}^{n+1}),$$

$$422 \quad (5.3b) \quad b_{1,j}^{n+1} = b_{1,j}^n - \bar{c} \Delta t (b_{1,j}^n - b_{1,j-1}^n) / \Delta y + i \bar{c} k_x \Delta t (a_{2,j}^{n+1} - d_j^{n+1}),$$

$$423 \quad (5.3c) \quad a_{2,j}^{n+1} = a_{2,j}^n + \bar{c} \Delta t (a_{2,j+1}^n - a_{2,j}^n) / \Delta y + i \bar{c} k_x \Delta t b_{1,j}^{n+1},$$

$$424 \quad (5.3d) \quad b_{2,j}^{n+1} = b_{2,j}^n - \bar{c} \Delta t (b_{2,j}^n - b_{2,j-1}^n) / \Delta y - i \bar{c} k_x \Delta t a_{1,j}^{n+1},$$

$$425 \quad (5.3e) \quad d_j^{n+1} = d_j^n,$$

427 where, for example,  $a_{1,j}^n \approx a_1(y_j, t^n)$  with  $y_j = j\Delta y$ . The grid in the  $y$ -direction is  
 428 collocated about the interface at  $y = 0$ . The terms involving transverse derivatives  
 429 are treated implicitly to stabilize the pseudo-spectral approximation. For reference,  
 430 the solid velocity and stress are related to the characteristic variables by  $\bar{v}_{m,j}^n =$   
 431  $\frac{1}{2\bar{z}}(a_{m,j}^n - b_{m,j}^n)$ ,  $\bar{\sigma}_{m2,j}^n = \frac{1}{2}(a_{m,j}^n + b_{m,j}^n)$ ,  $m = 1, 2$ . We assume bounded solutions  
 432 of (5.3) as  $y_j \rightarrow -\infty$ .

433 **5.3. Interface coupling.** We explore the stability of partitioned schemes for  
 434 the model problem that use different interface coupling approaches. For any of the  
 435 approaches, corresponding to the AMP, TP, and ATP schemes, the discrete equations  
 436 require a certain number of boundary conditions at the interface. For example, the  
 437 evolution of the fluid equations in (5.2) require three boundary conditions on the  
 438 interface,  $y = 0$ , to determine the interface velocity and pressure. Similarly, the  
 439 evolution of the solid equations in (5.3) require two boundary conditions at  $y = 0$   
 440 corresponding to the two incoming characteristic variables.

441 We first describe the coupling based on the AMP interface conditions given in  
 442 Section 3. We assume the fluid and solid solutions are known at time  $t^n$ . The solid  
 443 variables are advanced first to  $t^{n+1}$  on grid points  $j = 0, -1, -2, \dots$  using the evolution  
 444 equations in (5.3). The solid interface velocity and stress are computed using

$$445 \quad \bar{v}_{m,0}^{n+1} = \frac{1}{2\bar{z}}(a_{m,0}^{n+1} - b_{m,0}^{n+1}), \quad \bar{\sigma}_{m2,0}^{n+1} = \frac{1}{2}(a_{m,0}^{n+1} + b_{m,0}^{n+1}), \quad m = 1, 2.$$

447 The fluid velocity is advanced using (5.2a)–(5.2b). Two boundary conditions are  
 448 required at  $y = 0$  to obtain the fluid velocity at  $t^{n+1}$ . The condition on the outgoing  
 449 solid tangential characteristic in (3.1b) becomes

$$450 \quad (5.4) \quad \mu (ik_x v_2^{n+1} + \partial_y v_1^{n+1}) - \bar{z} v_1^{n+1} = \bar{\sigma}_{12,0}^{n+1} - \bar{z} \bar{v}_{1,0}^{n+1}, \quad y = 0.$$

452 The normal component of the velocity is projected to obtain the proper limiting  
 453 behaviors for heavy and light solids. This condition, taken from (3.8), reduces to

$$454 \quad (5.5) \quad v_2^{n+1} = \frac{z_f}{z_f + \bar{z}} V^p(v_2^{n+1}) + \frac{\bar{z}}{z_f + \bar{z}} \bar{v}_{2,0}^{n+1}, \quad y = 0,$$

456 where the fluid impedance is given by  $z_f = \rho/(k_x \Delta t) + 2\mu k_x$ , according to the deriva-  
 457 tion in Section 4. The predicted velocity,  $V^p(v_2^{n+1})$ , in (5.5) is given by

$$458 \quad (5.6) \quad V^p(v_2^{n+1}) = v_2^n - \frac{\Delta t}{\rho} \partial_y p^n - \nu \Delta t (k_x^2 v_2^{n+1} + ik_x \partial_y v_1^{n+1}), \quad y = 0.$$

460 This definition is analogous to the definition for  $\mathbf{V}^p(\mathbf{v}_i^{(p)})$  in Section 3, but with the  
 461 substitution  $\partial_y v_2^{n+1} = -ik_x v_1^{n+1}$  noting that  $\nabla \cdot \mathbf{v} = 0$  on the boundary. The pressure  
 462 is updated using (5.2c) along with the AMP pressure condition described in (3.4).  
 463 For the present scheme, this condition reduces to

$$464 \quad -p^{n+1} + \frac{\bar{z}\Delta t}{\rho} \partial_y p^{n+1} = \bar{\sigma}_{22,0}^{n+1} + 2ik_x \mu v_1^{n+1} - \bar{z}\Delta t [\dot{\bar{v}}_{2,0}^{n+1}$$

$$465 \quad (5.7) \quad + \nu (k_x^2 v_2^{n+1} + ik_x \partial_y v_1^{n+1})], \quad y = 0,$$

467 again using  $\partial_y v_2^{n+1} = -ik_x v_1^{n+1}$ . The acceleration of the solid on the interface,  
 468 denoted by  $\dot{\bar{v}}_{2,0}^{n+1}$  in (5.7), is taken to be  $\dot{\bar{v}}_{2,0}^{n+1} = (\bar{v}_{2,0}^{n+1} - \bar{v}_{2,0}^n)/\Delta t$ . After solving for

469 the fluid velocity and pressure, interface quantities from the fluid are obtained using

$$\begin{aligned}
470 \quad & v_{m,f}^{n+1} = v_m^{n+1}, \quad m = 1, 2, & p_f^{n+1} &= p^{n+1}, \\
471 \quad & \sigma_{12,f}^{n+1} = \mu (\partial_y v_1^{n+1} + ik_x v_2^{n+1}), & \sigma_{22,f}^{n+1} &= -p^{n+1} + 2\mu \partial_y v_2^{n+1},
\end{aligned}$$

473 where all fluid quantities on the right-hand side are evaluated at  $y = 0$ . The interface  
474 velocity and traction are projected from fluid and solid values using (3.6). These  
475 projections reduce to

$$476 \quad (5.8a) \quad v_m^I = \frac{z_f}{z_f + \bar{z}} v_{m,f}^{n+1} + \frac{\bar{z}}{z_f + \bar{z}} \bar{v}_{m,0}^{n+1} + \frac{1}{z_f + \bar{z}} (\bar{\sigma}_{m2,0}^{n+1} - \sigma_{m2,f}^{n+1}),$$

$$477 \quad (5.8b) \quad \sigma_{m2}^I = \frac{z_f^{-1}}{z_f^{-1} + \bar{z}^{-1}} \sigma_{m2,f}^{n+1} + \frac{\bar{z}^{-1}}{z_f^{-1} + \bar{z}^{-1}} \bar{\sigma}_{m2,0}^{n+1} + \frac{1}{z_f^{-1} + \bar{z}^{-1}} (\bar{v}_{m,0}^{n+1} - v_{m,f}^{n+1}),$$

479 where  $m = 1, 2$ . Finally, the ghost points at  $j = 1$  for the incoming solid characteristics  
480 are set using

$$481 \quad (5.9) \quad a_{m,1}^{n+1} = \sigma_{m2}^I + \bar{z} v_m^I, \quad m = 1, 2,$$

483 which is a first-order accurate approximation (consistent with the order of accuracy  
484 of the upwind scheme).

485 We next consider the coupling conditions for the TP and ATP schemes. These  
486 conditions can be obtained from the coupling conditions for the AMP scheme in the  
487 limits of heavy ( $\bar{z} \rightarrow \infty$ ) and light ( $\bar{z} \rightarrow 0$ ) solids. For the AMP algorithm, the  
488 fluid velocity and pressure conditions are given in (5.4), (5.5) and (5.7), while the  
489 final interface values are defined by the projections in (5.8). For the TP algorithm  
490 ( $\bar{z} \rightarrow \infty$ ), the AMP conditions in (5.4) and (5.5) reduce to Dirichlet conditions on the  
491 fluid velocity given by

$$492 \quad v_m^{n+1} = \bar{v}_{m,0}^{n+1}, \quad y = 0, \quad m = 1, 2.$$

494 The pressure condition in (5.7) becomes a Neumann condition given by

$$495 \quad (5.10) \quad \partial_y p^{n+1} = -\dot{\bar{v}}_{2,0}^{n+1} - \nu (k_x^2 v_2^{n+1} + ik_x \partial_y v_1^{n+1}), \quad y = 0.$$

497 For the TP scheme, the interface velocity is taken to be the solid velocity,  $v_m^I = \bar{v}_{m,0}^{n+1}$ ,  
498 and the interface traction is taken to be the fluid traction,  $\sigma_{m2}^I = \sigma_{m2,f}^{n+1}$ ,  $m = 1, 2$ .

499 For the ATP scheme, we consider the light-solid limit ( $\bar{z} \rightarrow 0$ ) of the AMP  
500 conditions. In this limit, the condition on the outgoing solid tangential characteristic  
501 in (5.4) reduces to a Neumann condition for the velocity given by

$$502 \quad (5.11) \quad \mu (ik_x v_2^{n+1} + \partial_y v_1^{n+1}) = \bar{\sigma}_{12,0}^{n+1}, \quad y = 0,$$

504 while the condition in (5.5) becomes

$$505 \quad (5.12) \quad v_2^{n+1} = V^p(v_2^{n+1}), \quad y = 0,$$

507 where  $V^p(v_2^{n+1})$  is given by (5.6). Using (5.2b), it can be shown that the condition  
508 in (5.12) can be replaced by

$$509 \quad (5.13) \quad \partial_y (ik_x v_1^{n+1} + \partial_y v_2^{n+1}) = 0, \quad y = 0,$$

511 which is equivalent to setting the fluid velocity to be divergence-free on the interface.  
 512 For the ATP scheme, the pressure condition in (5.7) reduces to

$$513 \quad (5.14) \quad -p^{n+1} + 2\mu\partial_y v_2^{n+1} = \bar{\sigma}_{22,0}^{n+1}, \quad y = 0.$$

515 For the ATP scheme, the interface velocity is taken to be the fluid velocity,  $v_m^I = v_{m,f}^{n+1}$ ,  
 516 and the interface traction is taken to be the solid traction,  $\sigma_{m2}^I = \bar{\sigma}_{m2,0}^{n+1}$ ,  $m = 1, 2$ .

517 **5.4. Stability analysis.** In order to assess the stability of the AMP, TP and  
 518 ATP schemes, we search for normal mode solutions to the discrete evolution equations.  
 519 In the fluid, solutions are of the form

$$520 \quad (5.15) \quad v_m^n(y) = A^n \tilde{v}_m(y), \quad p^n(y) = A^n \tilde{p}(y), \quad m = 1, 2,$$

522 where  $A$  is an amplification factor. Note that while it is not necessary to assume that  
 523 the amplification factors for velocity and pressure are the same in their initial forms,  
 524 the condition that these forms satisfy the momentum equations in (5.2a)–(5.2b) would  
 525 immediately imply that amplification factors are equal. Substituting (5.15) into (5.2)  
 526 and integrating gives

$$527 \quad (5.16a) \quad \tilde{v}_1(y) = v_{1,f}^0 e^{-\gamma|k_x|y} - \frac{ip_f^0}{\mu k_x A(\gamma^2 - 1)} \left( e^{-|k_x|y} - e^{-\gamma|k_x|y} \right),$$

$$528 \quad (5.16b) \quad \tilde{v}_2(y) = v_{2,f}^0 e^{-\gamma|k_x|y} + \frac{p_f^0}{\mu |k_x| A(\gamma^2 - 1)} \left( e^{-|k_x|y} - e^{-\gamma|k_x|y} \right),$$

$$529 \quad (5.16c) \quad \tilde{p}(y) = p_f^0 e^{-|k_x|y},$$

531 where  $\mu = \rho\nu$ ,  $\gamma = \sqrt{1 + (1 - 1/A)/\Lambda}$  and  $\Lambda = \nu k_x^2 \Delta t$ . Here,  $\Lambda$  represents the viscous  
 532 CFL number and we have imposed boundedness of the solution in (5.16) as  $y \rightarrow \infty$ .  
 533 The constants,  $v_{m,f}^0$  and  $p_f^0$ , are obtained by imposing the appropriate boundary  
 534 conditions at  $y = 0$ , namely, (5.4), (5.5) and (5.7) for the case of the AMP scheme.  
 535 For the TP scheme, the three constraints are the two boundary conditions for the  
 536 components of the velocity in (5.11) and the condition on the pressure in (5.10), while  
 537 the ATP scheme uses the boundary conditions in (5.11), (5.13) and (5.14).

538 Having found solutions for the velocity and pressure of the fluid, these solutions  
 539 can be used (along with the appropriate boundary conditions at  $y = 0$  for the AMP,  
 540 TP or ATP coupling) to eliminate the fluid variables on the boundary in favor of the  
 541 solid variables. The issue of stability, then, reduces to examining the behavior of the  
 542 evolution equations for the solid with the appropriate boundary conditions. Solutions  
 543 of these evolution equations are sought in the form

$$544 \quad (5.17) \quad \mathbf{a}_j^n = \phi^j A^n \tilde{\mathbf{r}}, \quad \mathbf{a}_j^n = [a_{1,j}^n, b_{1,j}^n, a_{2,j}^n, b_{2,j}^n, d_j^n]^T,$$

545 where  $\phi$  is a spatial eigenvalue and  $\tilde{\mathbf{r}}$  is a constant eigenvector. The scheme is said to  
 546 be weakly stable if there are no non-trivial solutions with  $|A| > 1$ . Our strategy for  
 547 determining regions of stability will be to search for unstable modes with  $|A| > 1$ , and  
 548 then identify regions of the parameter space where no non-trivial solutions exist. To  
 549 do this, we begin by finding the general solution for the spatial grid functions satisfying  
 550 the discrete equations and regularity condition as  $j \rightarrow -\infty$ , assuming  $|A| > 1$ . We  
 551 then apply the conditions at the interface to determine whether non-trivial solutions  
 552 exist.

553 Substitution of the normal mode ansatz (5.17) into the evolution equations for the  
 554 solid in (5.3), yields a  $5 \times 5$  homogeneous system of the form  $\mathcal{F}(\phi)\tilde{\mathbf{r}} = 0$ . Nontrivial  
 555 solutions exist if

$$556 \quad (5.18) \quad f(\phi) \stackrel{\text{def}}{=} \det(\mathcal{F}(\phi)) = (1 - A) (\eta(\phi)\eta(1/\phi) + (A\lambda_x)^2)^2 = 0,$$

558 where  $\eta(\phi) \stackrel{\text{def}}{=} 1 - A + \lambda_y(\phi - 1)$ ,  $\lambda_x \stackrel{\text{def}}{=} \bar{c}|k_x|\Delta t$  and  $\lambda_y \stackrel{\text{def}}{=} \bar{c}\Delta t/\Delta y$ . Since we seek  
 559 unstable modes with  $|A| > 1$ , the determinant condition in (5.18) is satisfied only  
 560 when  $\eta(\phi)\eta(1/\phi) = -(A\lambda_x)^2$ . This leads to the roots given by

$$561 \quad \phi_{\pm} = \xi \pm \sqrt{\xi^2 - 1}, \quad \xi = 1 - \frac{(A\lambda_x)^2 + (1 - A)^2}{2\lambda_y(1 - A - \lambda_y)}.$$

563 Note that the product of the roots is equal to one (i.e.  $\phi_+\phi_- = 1$ ). Since we are  
 564 searching for solutions that are bounded as  $j \rightarrow -\infty$ , we are only interested in the  
 565 root with modulus greater than one.

566 **Lemma:** *If  $|A| > 1$  and if  $\lambda_x$  and  $\lambda_y$  are chosen to satisfy a CFL condition, then*  
 567 *there is precisely one root, either  $\phi_+$  or  $\phi_-$ , denoted by  $\phi_*$  that has magnitude strictly*  
 568 *greater than one, i.e.  $|\phi_*| > 1$ .*

569 This result follows from a similar argument to that given in [18]. We first consider the  
 570 scheme applied to the pure initial-value problem (Cauchy problem). Setting  $\phi = e^{i\vartheta}$   
 571 in (5.18), we determine a region of the  $(\lambda_x, \lambda_y)$  plane for which  $|A| \leq 1$  for all  $\vartheta \in$   
 572  $[0, 2\pi]$ . This stable region is found numerically, and it includes a region satisfying a  
 573 reasonable CFL restriction, namely  $\lambda_x^2 + \lambda_y^2 \leq 1$  (see [25]). Next, since  $|A| \leq 1$  when  
 574  $|\phi| = 1$ , we have that  $|\phi| \neq 1$  when  $|A| > 1$ . Thus, if  $|A| > 1$  and if  $(\lambda_x, \lambda_y)$  remains  
 575 within the CFL restriction, then  $\phi$  cannot cross the unit circle,  $|\phi| = 1$ , as  $(\lambda_x, \lambda_y)$   
 576 vary. It is therefore only necessary to prove that the lemma holds for one set of  
 577 parameters. For  $\lambda_x = 0$ , the discretization reduces to four uncoupled upwind schemes  
 578 for linear advection. In this case, equation (5.18) is equivalent to  $\eta(\phi)\eta(1/\phi) = 0$ ,  
 579 which has solutions  $\phi_+ = (A - 1 + \lambda_y)/\lambda_y$  and  $\phi_- = 1/\phi_+$ . When  $|A| > 1$  and  
 580  $\lambda_y \in (0, 1]$ ,  $|\phi_+| > 1$  and therefore  $\phi_* = \phi_+$ . Thus, the condition holds for all  $(\lambda_x, \lambda_y)$   
 581 provided the CFL condition is satisfied.

582 The two eigenvectors associated with  $\phi = \phi_*$  lead to bounded solutions given by

$$583 \quad a_{1,j}^n = k_1 \text{sgn}(k_x) \frac{\eta(1/\phi_*)}{iA\lambda_x} \phi_*^j A^n, \quad b_{1,j}^n = -k_2 \text{sgn}(k_x) \frac{\eta(\phi_*)}{iA\lambda_x} \phi_*^j A^n,$$

$$584 \quad a_{2,j}^n = k_2 \phi_*^j A^n, \quad b_{2,j}^n = k_1 \phi_*^j A^n,$$

586 where  $k_1$  and  $k_2$  are constants to be determined by the two interface conditions in (5.9).  
 587 The application of these interface conditions leads to a  $2 \times 2$  homogeneous system  
 588 of the form  $\mathcal{G}(A)\mathbf{k} = 0$ . Solutions for the amplification factor  $A$  are roots of the  
 589 transcendental equation given by

$$590 \quad g(A) \stackrel{\text{def}}{=} \det(\mathcal{G}(A)) = 0.$$

592 These roots depend on the choice of interface coupling (AMP, TP or ATP) and four  
 593 dimensionless parameters  $(\Lambda, Z, \lambda_x, \lambda_y)$ , where  $Z \stackrel{\text{def}}{=} \mu|k_x|/\bar{z}$ . (Further details are  
 594 given in [25].)

595 Proving stability of the partitioned scheme for a choice of the interface coupling  
 596 and dimensional parameters is equivalent to showing that no roots of  $g(A) = 0$  exist



597 such that  $|A| > 1$ . The number of roots with  $|A| > 1$  can be assessed using the  
 598 argument principle. Define

$$599 \quad \mathcal{P} \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{G'(\zeta)}{G(\zeta)} d\zeta, \quad G(\zeta) = g(1/\zeta). \\ 600$$

601 There are branch points of  $G(\zeta)$  in the region  $|\zeta| > 1$ , and a single-valued branch  
 602 of  $G(\zeta)$  can be defined so that its branch cuts lie outside the unit disk. Using this  
 603 definition, the only singularity of  $G(\zeta)$  in the region  $|\zeta| \leq 1$  is a pole of order 2  
 604 at the origin, and thus  $\mathcal{P} = N - 2$ , where  $N$  corresponds to the number of roots of  $g(A)$   
 605 with  $|A| > 1$ .

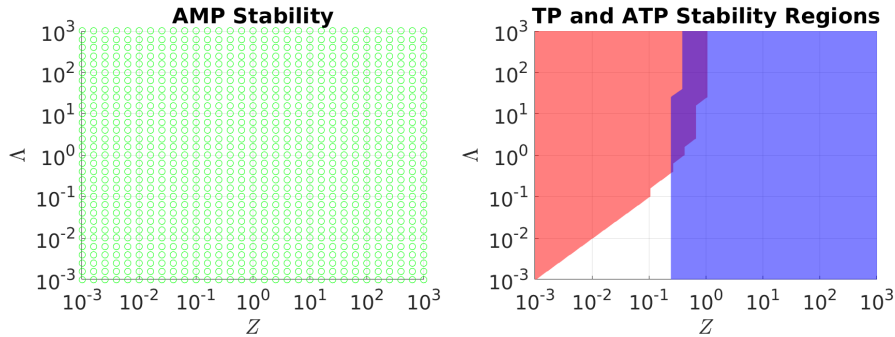


FIG. 2. Left: Green circles represent points for which the AMP algorithm is stable in the CFL region  $\lambda_x^2 + \lambda_y^2 \leq 1$ . Right: stability regions for the TP (red) and ATP (blue) schemes.

606 An analytic evaluation of the integral for  $\mathcal{P}$  is unavailable, and so we consider  
 607 a numerical evaluation. The four-dimensional parameter space  $(Z, \Lambda, \lambda_x, \lambda_y)$  is dis-  
 608 cretized on a  $31 \times 31 \times 20 \times 20$  array. The parameters  $Z$  and  $\Lambda$  are equally spaced on a  
 609 logarithmic scale on the interval  $[10^{-3}, 10^3]$ , while  $\lambda_x$  and  $\lambda_y$  are equally spaced on the  
 610 interval  $[0.05, 0.95]$ . At each grid point,  $\mathcal{P}$  is computed numerically with  $|\mathcal{P} + 2| \leq \delta$   
 611 corresponding to stability, where  $\delta$  is a small parameter taken to be  $10^{-5}$ . The results  
 612 of this computation are shown in Figure 2 for the AMP, TP and ATP schemes. A  
 613 grid point in the  $(\Lambda, Z)$  plane is marked as stable if the computations of  $\mathcal{P}$  for all  
 614 values of  $\lambda_x$  and  $\lambda_y$  in the search region are stable. The point is marked as unstable  
 615 otherwise. The results shown in the left plot indicate that the AMP scheme applied  
 616 to the viscous model problem is stable for all points in the  $(\Lambda, Z)$  plane, whereas the  
 617 results shown in the right plot indicate that the TP and ATP schemes have large  
 618 regions of instability. For example, the region in red shows the stable region for the  
 619 TP scheme, which occurs for heavy solids ( $Z$  small) and coarser meshes ( $\Lambda$  large).  
 620 The stability region for the ATP scheme shown in blue corresponds to light solids  
 621 ( $Z$  large). The following theorem summarizes the results for the AMP scheme.

622 **Theorem:** *The AMP scheme applied to viscous model problem is weakly stable  $|A| \leq 1$*   
 623 *provided  $\lambda_x^2 + \lambda_y^2 \leq 1$ , which gives the usual CFL-type time-step restriction*

$$624 \quad \Delta t \leq \frac{1}{c} \left[ \frac{1}{\Delta y^2} + k_x^2 \right]^{1/2}. \\ 625$$

626 *This is a sufficient but not a necessary condition. The proof follows from the argument*  
 627 *principle and a numerical evaluation of  $\mathcal{P}$ .*

628 **6. Numerical results for an elastic piston.** We now present numerical results  
629 for two FSI problems to verify the accuracy and stability of the AMP scheme. The  
630 two FSI problems considered involve the interaction of a fluid column with an elastic  
631 piston. In the first problem, we examine longitudinal motion of the piston (which  
632 primarily involves added-mass effects), while transverse motion of the piston (which  
633 primarily involves added-damping effects) is considered in the second problem. Exact  
634 solutions are found for both FSI problems, and these are used to verify the accuracy  
635 and stability of the AMP algorithm for a range of the problem parameters.

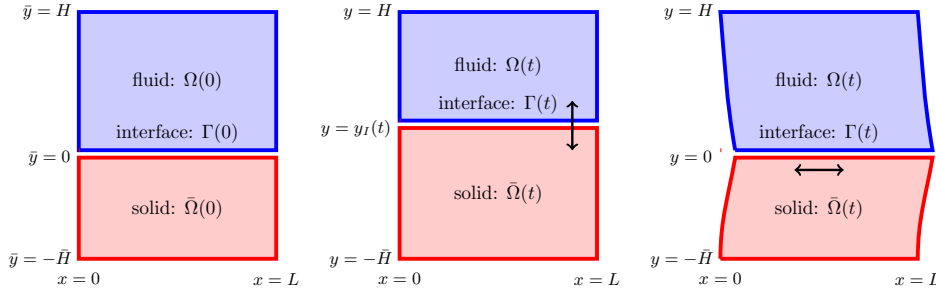


FIG. 3. FSI problem coupling an incompressible viscous fluid and an elastic piston: Configuration at  $t = 0$  (left), longitudinal motion for  $t > 0$  (middle) and shear motion for  $t > 0$  (right).

636 *Longitudinal motion of an elastic piston.* The geometry of the elastic piston problem  
637 is shown in Figure 3. The plot on the left shows the configuration at  $t = 0$ . The  
638 fluid occupies the physical domain between  $y = 0$  and  $y = H$  initially, while the solid  
639 lies in its reference domain between  $\bar{y} = -\bar{H}$  and  $\bar{y} = 0$ . It is assumed that there  
640 is no dependence in the  $x$ -direction so that the fluid-solid interface remains flat at a  
641 position  $y = y_I(t)$  as shown in the plot on the right. In the fluid domain,  $\Omega(t)$ , it is  
642 assumed that the horizontal component of velocity  $v_1$  is zero, and thus the vertical  
643 component  $v_2$  is a function of  $t$  alone according to the continuity equation.

644 Solutions to this FSI problem can be constructed for a specified motion of the  
645 fluid solid interface, see [25] for more details. We choose an interface position  $y_I(t)$   
646 that oscillates in the vertical direction with frequency  $\omega$  and an amplitude  $a$  given by

$$647 \quad y_I(t) = a \sin(\omega t), \quad a = 2\alpha \sin(\omega \bar{H} / \bar{c}_p).$$

648 Numerical results are obtained for the case  $H = 1, \rho = 1$  and  $\mu = 0.01$  for the  
649 fluid, and using  $\bar{H} = 0.5$  and  $\bar{\mu} = \bar{\lambda} = \bar{\rho} = \delta$  for the solid. The interface position is  
650 specified by  $a = 0.1$  and  $\omega = 2\pi$ . The density ratio,  $\bar{\rho}/\rho = \delta$ , is taken to be  $10^{-3}$ , 1 and  
651  $10^3$ , representing FSI problems with light, moderate and heavy solids, respectively.  
652 Numerical solutions are computed using the AMP algorithm on a two-dimensional  
653 rectangular configuration (as shown in Figure 3) with periodic boundary conditions  
654 taken at  $x = 0$  and  $x = L$  consistent with a one-dimensional solution. Table 1  
655 gives the maximum-norm errors for solutions computed using the AMP algorithm at  
656  $t_{\text{final}} = 0.6$  with grid resolutions  $h = 1/(20j)$  for  $j = 1, 2, 4, 8$ . The errors in the table  
657 indicate that the solution is converging at second-order accuracy.

658 *Transverse motion of an elastic piston.* Exact solutions can also be constructed  
659 for an FSI problem involving transverse motion of an elastic piston, see Figure 3.  
660 For this case, the vertical components of the fluid velocity and solid displacement are  
661 taken to be zero, while the corresponding horizontal components are assumed to be  
662 functions of  $y$  and  $t$  alone. As a result, the interface only moves horizontally so that

TABLE 1

Longitudinal motion of an elastic piston: Maximum-norm errors and convergence ratios of the numerical solution at  $t_{\text{final}} = 0.6$  computed using the AMP algorithm for  $\bar{\rho}/\rho = \delta = 10^3, 1$  and  $10^{-3}$ .

Heavy solid ( $\delta = 10^3$ ):										
$h$	$E^{(p)}$	$r$	$E^{(v)}$	$r$	$E^{(\bar{u})}$	$r$	$E^{(\bar{v})}$	$r$	$E^{(\bar{\sigma})}$	$r$
1/ 20	6.0e-04		5.9e-05		4.0e-05		5.9e-05		1.9e-01	
1/ 40	1.4e-04	4.2	1.7e-05	3.5	9.6e-06	4.2	1.7e-05	3.5	4.5e-02	4.2
1/ 80	3.4e-05	4.1	4.4e-06	3.8	2.3e-06	4.1	4.4e-06	3.8	1.1e-02	4.1
1/160	8.5e-06	4.1	1.1e-06	3.9	5.8e-07	4.1	1.1e-06	3.9	2.7e-03	4.1
Medium solid ( $\delta = 1$ ):										
$h$	$E^{(p)}$	$r$	$E^{(v)}$	$r$	$E^{(\bar{u})}$	$r$	$E^{(\bar{v})}$	$r$	$E^{(\bar{\sigma})}$	$r$
1/ 20	1.8e-05		4.9e-05		1.2e-05		4.9e-05		5.0e-05	
1/ 40	7.5e-06	2.4	1.2e-05	4.0	3.0e-06	4.2	1.2e-05	4.0	1.3e-05	3.7
1/ 80	2.3e-06	3.3	3.0e-06	4.0	7.1e-07	4.1	3.0e-06	4.0	3.6e-06	3.8
1/160	6.3e-07	3.6	7.4e-07	4.0	1.8e-07	4.1	7.4e-07	4.0	9.2e-07	3.9
Light solid ( $\delta = 10^{-3}$ ):										
$h$	$E^{(p)}$	$r$	$E^{(v)}$	$r$	$E^{(\bar{u})}$	$r$	$E^{(\bar{v})}$	$r$	$E^{(\bar{\sigma})}$	$r$
1/ 20	8.0e-07		6.5e-07		3.3e-06		2.4e-05		1.3e-07	
1/ 40	2.4e-07	3.3	1.6e-07	4.0	5.3e-07	6.3	4.2e-06	5.7	3.4e-08	3.9
1/ 80	6.6e-08	3.7	4.1e-08	4.0	8.9e-08	5.9	8.3e-07	5.0	8.8e-09	3.9
1/160	1.7e-08	3.8	1.0e-08	4.0	2.3e-08	3.8	1.8e-07	4.5	2.2e-09	4.0

663  $y_I(t) = y_I(0) = 0$ , and the solid reference coordinate  $\bar{y}$  is equivalent to the physical  
664 coordinate  $y$ .

665 For this problem, the equations governing the horizontal components of the fluid  
666 velocity and solid displacement reduce to  $v_{1,t} = \nu v_{1,yy}$ , for  $0 < y < H$ , and  $\bar{u}_{1,tt} =$   
667  $\bar{c}_s^2 \bar{u}_{1,yy}$ , for  $-H < y < 0$ , and there are time periodic solutions with  $v_1(y, t) =$   
668  $\hat{v}_1(y) e^{i\omega t}$  and  $\bar{u}_1(y, t) = \hat{\bar{u}}_1(y) e^{i\omega t}$  for certain values of the eigenvalue  $\omega$  (see [25] for  
669 further details).

670 Solutions to this problem are computed for selected values of  $\omega$  (as noted in  
671 Table 2) for  $H = 1$ ,  $\bar{H} = 0.5$ ,  $\rho = 1$  and  $\mu = 0.1$ , and for different values of  $\delta = \bar{\rho} =$   
672  $\bar{\mu} = \bar{\lambda}$ . The magnitude of the interface displacement in the  $x$ -direction at  $t = 0$  is  
673 taken as  $\bar{u}_0 = 0.1$ . Table 2 gives the maximum-norm errors for solutions computed  
674 using the AMP algorithm. The results are presented for solutions at  $t_{\text{final}} = 0.3$  using  
675 grid resolutions  $h = 1/(20j)$ , for  $j = 1, 2, 4, 8$ . The errors in the table indicate that  
676 the solution is converging at second-order accuracy.

677 **7. Conclusions.** A stable added-mass partitioned (AMP) algorithm was devel-  
678 oped for fluid-structure interaction problems involving viscous incompressible fluids  
679 and compressible elastic solids. The new algorithm is stable, without sub-time-step  
680 iterations, for both heavy and very light solids thus effectively suppresses both added-  
681 mass and added-damping effects. The fluid is advanced using a fractional-step IMEX  
682 scheme with the viscous terms treated implicitly. Key elements of the new AMP  
683 scheme are a Robin interface condition for the pressure and an impedance-weighted  
684 interface projection based on a new form for the fluid impedance. The fluid impedance  
685 is derived from the analysis of a carefully chosen FSI model problem. Stability of the  
686 AMP scheme is analyzed for a related model problem. A set of *elastic-piston* bench-  
687 mark problems was developed to verify the stability and accuracy of the AMP scheme.  
688 These solutions are exact and include finite interface deformations either normal or  
689 tangential to the surface.

TABLE 2

Transverse motion of an elastic piston: Maximum-norm errors and convergence ratios of the numerical solution at  $t_{\text{final}} = 0.3$  computed using the AMP algorithm for  $\bar{\rho}/\rho = \delta = 10^3, 1$  and  $10^{-3}$ .

Heavy solid ( $\delta = 10^3, \omega = 3.141 + i 7.930 \cdot 10^{-4}$ ):										
$h$	$E^{(p)}$	$r$	$E^{(v)}$	$r$	$E^{(\bar{u})}$	$r$	$E^{(\bar{v})}$	$r$	$E^{(\bar{\sigma})}$	$r$
1/ 20	6.0e-04		5.9e-05		4.0e-05		5.9e-05		1.9e-01	
1/ 40	1.4e-04	4.2	1.7e-05	3.5	9.6e-06	4.2	1.7e-05	3.5	4.5e-02	4.2
1/ 80	3.4e-05	4.1	4.4e-06	3.8	2.3e-06	4.1	4.4e-06	3.8	1.1e-02	4.1
1/160	8.5e-06	4.1	1.1e-06	3.9	5.8e-07	4.1	1.1e-06	3.9	2.7e-03	4.1
Medium solid ( $\delta = 1, \omega = 2.351 + i 5.433 \cdot 10^{-1}$ ):										
$h$	$E^{(p)}$	$r$	$E^{(v)}$	$r$	$E^{(\bar{u})}$	$r$	$E^{(\bar{v})}$	$r$	$E^{(\bar{\sigma})}$	$r$
1/ 20	1.8e-05		4.9e-05		1.2e-05		4.9e-05		5.0e-05	
1/ 40	7.5e-06	2.4	1.2e-05	4.0	3.0e-06	4.2	1.2e-05	4.0	1.3e-05	3.7
1/ 80	2.3e-06	3.3	3.0e-06	4.0	7.1e-07	4.1	3.0e-06	4.0	3.6e-06	3.8
1/160	6.3e-07	3.6	7.4e-07	4.0	1.8e-07	4.1	7.4e-07	4.0	9.2e-07	3.9
Light solid ( $\delta = 10^{-3}, \omega = 6.285 + i 1.784 \cdot 10^{-3}$ ):										
$h$	$E^{(p)}$	$r$	$E^{(v)}$	$r$	$E^{(\bar{u})}$	$r$	$E^{(\bar{v})}$	$r$	$E^{(\bar{\sigma})}$	$r$
1/ 20	8.0e-07		6.5e-07		3.3e-06		2.4e-05		1.3e-07	
1/ 40	2.4e-07	3.3	1.6e-07	4.0	5.3e-07	6.3	4.2e-06	5.7	3.4e-08	3.9
1/ 80	6.6e-08	3.7	4.1e-08	4.0	8.9e-08	5.9	8.3e-07	5.0	8.8e-09	3.9
1/160	1.7e-08	3.8	1.0e-08	4.0	2.3e-08	3.8	1.8e-07	4.5	2.2e-09	4.0

690 The present AMP algorithm assumes a linear elastic constitutive model for the  
691 finite deformation of the solid. This was done as a first step towards an extension to  
692 nonlinear hyperelastic models, such as neo-Hookean or Saint Venant-Kirchoff. Such  
693 an extension was considered previously in [4] for FSI problems involving inviscid  
694 compressible fluids. In that paper, the AMP interface conditions used a linearization  
695 of the nonlinear model locally about points along the interface, and so the linear elastic  
696 model considered here should provide useful information for an analogous extension  
697 to FSI problems coupling viscous incompressible fluids and nonlinear solids. Such an  
698 extension is planned for future work.

699

## REFERENCES

- 700 [1] D. APPELÖ, J. W. BANKS, W. D. HENSHAW, AND D. W. SCHWENDEMAN, *Numerical methods*  
701 *for solid mechanics on overlapping grids: Linear elasticity*, J. Comput. Phys., 231 (2012),  
702 pp. 6012–6050.
- 703 [2] S. BADIA, F. NOBILE, AND C. VERGARA, *Fluid–structure partitioned procedures based on Robin*  
704 *transmission conditions*, J. Comput. Phys., 227 (2008), pp. 7027–7051, [https://doi.org/  
705 http://dx.doi.org/10.1016/j.jcp.2008.04.006](https://doi.org/http://dx.doi.org/10.1016/j.jcp.2008.04.006).
- 706 [3] S. BADIA, F. NOBILE, AND C. VERGARA, *Robin-Robin preconditioned Krylov methods for fluid–*  
707 *structure interaction problems*, CMAME, 198 (2009), pp. 2768–2784.
- 708 [4] J. W. BANKS, W. D. HENSHAW, A. KAPILA, AND D. W. SCHWENDEMAN, *An added-mass parti-*  
709 *tioned algorithm for fluid-structure interactions of compressible fluids and nonlinear solids*,  
710 J. Comput. Phys., 305 (2016), pp. 1037–1064.
- 711 [5] J. W. BANKS, W. D. HENSHAW, AND D. W. SCHWENDEMAN, *Deforming composite grids for*  
712 *solving fluid structure problems*, J. Comput. Phys., 231 (2012), pp. 3518–3547, [https://  
713 doi.org/10.1016/j.jcp.2011.12.034](https://doi.org/10.1016/j.jcp.2011.12.034).
- 714 [6] J. W. BANKS, W. D. HENSHAW, AND D. W. SCHWENDEMAN, *An analysis of a new stable*  
715 *partitioned algorithm for FSI problems. Part I: Incompressible flow and elastic solids*, J.  
716 Comput. Phys., 269 (2014), pp. 108–137.
- 717 [7] J. W. BANKS, W. D. HENSHAW, AND D. W. SCHWENDEMAN, *An analysis of a new stable*  
718 *partitioned algorithm for FSI problems. Part II: Incompressible flow and structural shells*,

- 719 J. Comput. Phys., 268 (2014), pp. 399–416.
- 720 [8] J. W. BANKS, W. D. HENSHAW, D. W. SCHWENDEMAN, AND Q. TANG, *A stable partitioned*  
721 *FSI algorithm for rigid bodies and incompressible flow. Part I: Model problem analysis.*,  
722 J. Comput. Phys., 343 (2017), pp. 432–468.
- 723 [9] J. W. BANKS, W. D. HENSHAW, D. W. SCHWENDEMAN, AND Q. TANG, *A stable partitioned*  
724 *FSI algorithm for rigid bodies and incompressible flow. Part II: General formulation*, J.  
725 Comput. Phys., 343 (2017), pp. 469–500.
- 726 [10] J. W. BANKS, W. D. HENSHAW, D. W. SCHWENDEMAN, AND Q. TANG, *A stable partitioned FSI*  
727 *algorithm for rigid bodies and incompressible flow in three dimensions*, J. Comput. Phys.,  
728 373 (2018), pp. 455–492.
- 729 [11] J. W. BANKS AND B. SJÖGREEN, *A normal mode stability analysis of numerical interface*  
730 *conditions for fluid/structure interaction*, Commun. Comput. Phys., 10 (2011), pp. 279–  
731 304.
- 732 [12] S. BASTING, A. QUAINI, S. ČANIĆ, AND R. GLOWINSKI, *Extended ALE method for fluid-*  
733 *structure interaction problems with large structural displacements*, Journal of Computa-  
734 tional Physics, 331 (2017), pp. 312 – 336, [https://doi.org/https://doi.org/10.1016/j.jcp.](https://doi.org/https://doi.org/10.1016/j.jcp.2016.11.043)  
735 [2016.11.043](https://doi.org/https://doi.org/10.1016/j.jcp.2016.11.043).
- 736 [13] M. A. FERNÁNDEZ AND M. LANDAJUELA, *Fully decoupled time-marching schemes for incom-*  
737 *pressible fluid/thin-walled structure interaction*, Rapport de recherche RR-8425, INRIA,  
738 Jan. 2014.
- 739 [14] M. A. FERNÁNDEZ, J. MULLAERT, AND M. VIDRASCU, *Explicit Robin-Neumann schemes for*  
740 *the coupling of incompressible fluids with thin-walled structures*, Comput. Method. Appl.  
741 Mech. Engrg., 267 (2013), pp. 566–593.
- 742 [15] M. A. FERNÁNDEZ, J. MULLAERT, AND M. VIDRASCU, *Generalized Robin-Neumann explicit*  
743 *coupling schemes for incompressible fluid–structure interaction: stability analysis and nu-*  
744 *merics*, Int. J. Numer. Meth. Eng., (2014), <https://doi.org/10.1002/nme.4785>.
- 745 [16] L. GERARDO-GIORDA, F. NOBILE, AND C. VERGARA, *Analysis and optimization of Robin-Robin*  
746 *partitioned procedures in fluid–structure interaction problems*, SIAM J. Numer. Anal., 48  
747 (2010), pp. 2091–2116.
- 748 [17] W. D. HENSHAW AND N. A. PETERSSON, *A split-step scheme for the incompressible Navier-*  
749 *Stokes equations*, in Numerical Simulation of Incompressible Flows, M. M. Hafez, ed.,  
750 World Scientific, 2003, pp. 108–125.
- 751 [18] H. O. KREISS, *Difference approximations for the initial-boundary value problem for hyperbolic*  
752 *differential equations*, Numerical Solutions of Nonlinear Differential Equations, (1996),  
753 pp. 141–166.
- 754 [19] U. KÜTTLER AND W. A. WALL, *Fixed-point fluid–structure interaction solvers with dynamic*  
755 *relaxation*, Computational Mechanics, 43 (2008), pp. 61–72, [https://doi.org/10.1007/](https://doi.org/10.1007/s00466-008-0255-5)  
756 [s00466-008-0255-5](https://doi.org/10.1007/s00466-008-0255-5).
- 757 [20] L. LI, W. D. HENSHAW, J. W. BANKS, D. W. SCHWENDEMAN, AND G. A. MAIN, *A stable*  
758 *partitioned FSI algorithm for incompressible flow and deforming beams*, J. Comput. Phys.,  
759 312 (2016), pp. 272–306.
- 760 [21] M. MEHL, B. UEKERMANN, H. BIJL, D. BLOM, B. GATZHAMMER, AND A. VAN ZUIJLEN, *Parallel*  
761 *coupling numerics for partitioned fluid-structure interaction simulations*, Computers &  
762 Mathematics with Applications, 71 (2016), pp. 869 – 891, [https://doi.org/https://doi.org/](https://doi.org/https://doi.org/10.1016/j.camwa.2015.12.025)  
763 [10.1016/j.camwa.2015.12.025](https://doi.org/https://doi.org/10.1016/j.camwa.2015.12.025).
- 764 [22] D. P. MOK, W. A. WALL, AND E. RAMM, *Accelerated iterative substructuring schemes for insta-*  
765 *tionary fluid structure interaction*, in Computational Fluid and Solid Mechanics, K. Bathe,  
766 ed., Elsevier, 2001, pp. 1325–1328.
- 767 [23] F. NOBILE, M. POZZOLI, AND C. VERGARA, *Inexact accurate partitioned algorithms for fluid-*  
768 *structure interaction problems with finite elasticity in haemodynamics*, Journal of Com-  
769 putational Physics, 273 (2014), pp. 598 – 617, [https://doi.org/http://dx.doi.org/10.1016/](https://doi.org/http://dx.doi.org/10.1016/j.jcp.2014.05.020)  
770 [j.jcp.2014.05.020](https://doi.org/http://dx.doi.org/10.1016/j.jcp.2014.05.020).
- 771 [24] D. A. SERINO, J. W. BANKS, W. D. HENSHAW, AND D. W. SCHWENDEMAN, *A stable added-*  
772 *mass partitioned (AMP) algorithm for elastic solids and incompressible flow*, preprint  
773 arXiv:1812.05208, 2019. submitted for publication.
- 774 [25] D. A. SERINO, J. W. BANKS, W. D. HENSHAW, AND D. W. SCHWENDEMAN, *A stable added-mass*  
775 *partitioned (AMP) algorithm for elastic solids and incompressible flow: Model problem*  
776 *analysis*, preprint arXiv:1812.03192, 2019. submitted for publication.
- 777 [26] Y. WANG, A. QUAINI, AND S. ČANIĆ, *A higher-order discontinuous Galerkin/arbitrary La-*  
778 *grangian Eulerian partitioned approach to solving fluid–structure interaction problems with*  
779 *incompressible, viscous fluids and elastic structures*, Journal of Scientific Computing, 76  
780 (2018), pp. 481–520, <https://doi.org/10.1007/s10915-017-0629-y>.